1

→ 1) X is compact iff for every filter $\mathcal{F}$ in X

$\operatorname{adh}\mathcal{F} \neq \emptyset$. $\operatorname{adh}\mathcal{F} =$

$$\operatorname{adh}\mathcal{F} = \bigcap_{F\in\mathcal{F}} \operatorname{cl} F$$

Show equiv. to Borel-Lebesgue condition $\{U_\alpha\}_{\alpha\in A}$

So assume B-L does not hold. So

given arbitrary cover, no finite subcover

exists.

Look @ $\left\{ X \setminus \bigcup_{i\in H} U_i : H \subset A \right\} = F$

they are : all closed

any 2 intersect

non-void (only finitely many taken away)

So collection is a filter base of closed sets.

filter $\langle \mathcal{F} \rangle$

by compactness

So $\mathrm{adh}\langle \mathcal{F} \rangle = \mathrm{adh}\, \mathcal{F} \neq \emptyset$

$$\exists\, x \in cl\left(\left\{X \setminus \bigcup_{i \in H} U_i : H \subset A\right\}\right)$$

but since is already closed ...

$$x \in X \setminus \bigcup_{i \in H} U_i \;,\; \forall H \subset A$$

$\underline{x \in U_i^c}$ $\forall i$ so also $\forall \kappa \in A$

FIP := Finite intersection Property

Given family of sets $\mathcal{F} \subset 2^{2^X}$

$\mathcal{F}$ has FIP if every finite subfamily has nonvoid $\bigcap$.

Ex: $[0, \frac{1}{n}]$ $n \in \mathbb{N}$

$(0, \frac{1}{n})$ $n \in \mathbb{N}$

<u>B-L</u> $\Rightarrow$ FIP

Given $\mathcal{F}$, arbitrary family of closed sets in X with FIP. Assume <u>FSOC</u> that $\bigcap \{F \in \mathcal{F}\} = \phi$. $\{F^c : F \in \mathcal{F}\}$

this is open cover of X

By B-L must have finite subcover

(4)

$$\bigcap_{i \in H} F_i^{*} = \phi.$$

Th^{m} If X is compact then every ultrafilter on X converges.

Given $\mathcal{B}$ ultrafilter in X. Claim:

$\mathcal{B}$ has FIP. (because of filter def^{n})

Also $\underline{\{cl\, B : B \in \mathcal{B}\}}$ has FIP.

By hypothesis $\bigcap_{B \in \mathcal{B}} cl\, B \neq \phi$.

$\Rightarrow x \in adh\, \mathcal{B}$ then $\mathcal{B} \to x$.

Tikhonov ~~Tychonoff~~ Th^m

If X_α is compact (rel. to τ_α) for $\alpha \in A$, arbitrary. Then $\prod_{\alpha \in A} X_\alpha$ is compact (rel. to product topology). And v.v.

Pf: Assume each X_α is τ_α-compact. Let M be <u>any</u> ultrafilter on $\prod X_\alpha$.

$\mathbb{R}^{\mathbb{R}}$

Consider projection $P_\alpha\left(\prod X_\alpha\right) = X_\alpha$

Suppose $P_\alpha(M) \to x_\alpha$

Claim $M \to \prod_{\alpha \in A} x_\alpha$ so $\prod X_\alpha$ compact.

$\rightarrow$ Banach-Alaoglu Th^m