

①

→ 1)  $X$  is compact iff for every filter  $\mathcal{F}$  in  $X$   
 $\text{adh } \mathcal{F} \neq \emptyset$ .  $\text{adh } \mathcal{F} = \bigcap_{F \in \mathcal{F}} F$

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Show equiv. to Borel-Lebesgue condition  
 $\{U_\alpha\}_{\alpha \in A}$

So assume B-L does not hold. So  
given arbitrary cover, no finite subcover  
exists.

$$\text{Look @ } \left\{ X - \bigcup_{i \in H} U_i : H \subset A \right\} = \mathcal{F}$$

they are : all closed

any 2 intersect

non-void (only finitely many taken away)

So collection is a filter<sup>base</sup> of closed sets.

filter  $\langle \mathcal{F} \rangle$

(2)

↳ by compactness

$$\text{So } \text{adh}\langle \mathcal{A} \rangle = \text{adh} \mathcal{A} \neq \emptyset$$

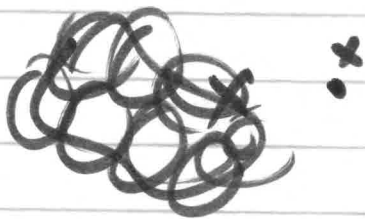
$$\exists x \in \mathcal{C} \left( X \setminus \bigcup_{i \in \mathcal{H}} U_i : \mathcal{H} \subset \mathcal{A} \right)$$

but since  $\mathcal{C}$  is already closed ...

$$x \in X \setminus \bigcup_{i \in \mathcal{H}} U_i, \quad \forall \mathcal{H} \subset \mathcal{A}$$

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$$\underline{x \in U_i} \quad \forall i \text{ so also } \forall x \in \mathcal{A}$$



(3)

FIP := Finite Intersection Property

Given family of sets  $\mathcal{F}_i \subset \mathcal{P}(X)$

$\mathcal{F}_i$  has FIP if every finite subfamily has nonvoid  $\cap$ .

Ex:  $[0, \frac{1}{n}]$   $n \in \mathbb{N}$

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B-L  $\Rightarrow$  FIP

Given  $\mathcal{F}_i$ , arbitrary family of closed sets

in  $X$  with FIP. Assume FSOC that

$\bigcap \{F \in \mathcal{F}_i\} = \emptyset$ .  $\{F^c : F \in \mathcal{F}_i\}$

$\uparrow$   
this is open cover  
of  $X$

By B-L must have finite subcover

$$\bigcap_{i \in \mathbb{N}} F_i^c = \emptyset. \quad \textcircled{A}$$

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Th<sup>y</sup> If  $X$  is compact then every ultrafilter on  $X$  converges.

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Given  $\mathcal{B}$  ultrafilter in  $X$ . Claim:

$\mathcal{B}$  has FIP. (because of filter def<sup>n</sup>)

Also  $\{ \mathcal{C} B : B \in \mathcal{B} \}$  has FIP.

By hypothesis  $\bigcap_{B \in \mathcal{B}} \mathcal{C} B \neq \emptyset$ .

$\Rightarrow x \in \text{adh } \mathcal{B}$  then  $\mathcal{B} \rightarrow x$ .

(5)

Tikhonov ~~Thm~~ Th<sup>m</sup>

If  $X_\alpha$  is compact (rel. to  $\tau_\alpha$ )  
for  $\alpha \in A$ , arbitrary. Then

$\prod_{\alpha \in A} X_\alpha$  is compact (rel. to product  
topology). And v.v.

Pf: Assume each  $X_\alpha$  is  $\tau_\alpha$ -compact.

Let  $M$  be any ultrafilter on  $\prod X_\alpha$ .

Consider projection  $P_\alpha(\prod X_\alpha) = X_\alpha$

Suppose  $P_\alpha(M) \rightarrow x_\alpha$

Claim  $M \rightarrow \prod_{\alpha \in A} X_\alpha$  so  $\prod X_\alpha$  compact.

→ Banach-Alaoglu Th<sup>m</sup>