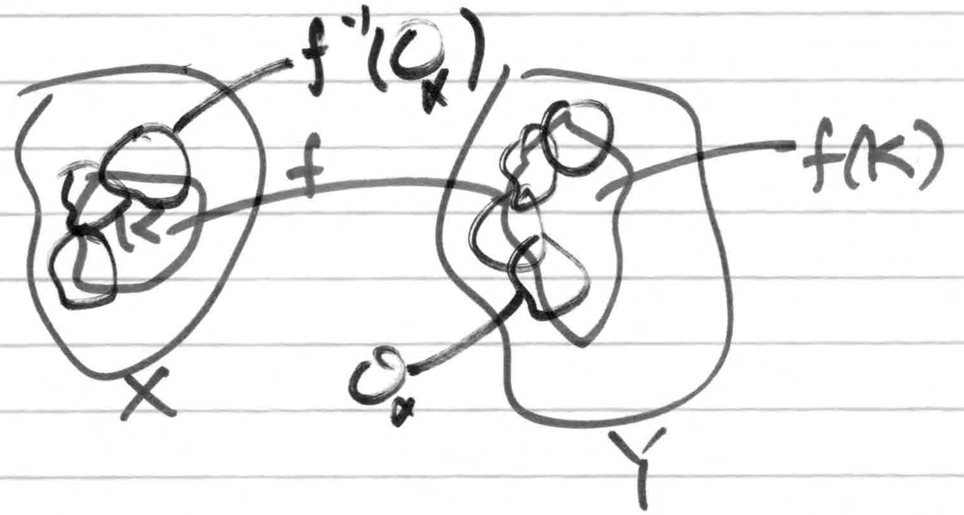
Th^m If $K \subset X$ is compact and $f: X \to Y$ is continuous, then $f(X) \subset Y$ is compact.

Pf:

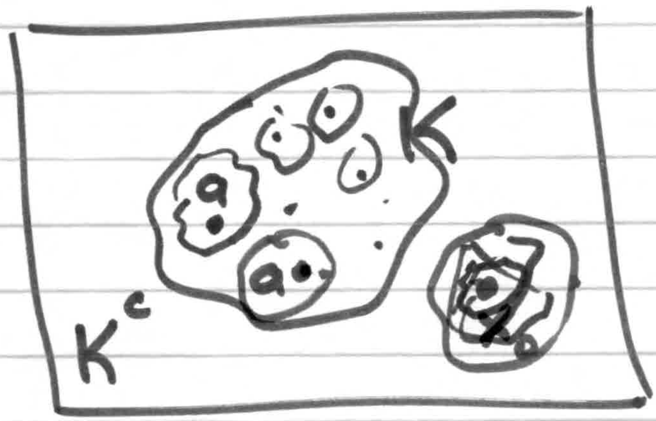


To show $f(K)$ compact assume $\{O_\alpha\}_{\alpha\in\Lambda}$ is opencover of $f(K)$, i.e. $f(K) \subset \bigcup_{\alpha\in\Lambda} O_\alpha$.

Consider $\{f^{-1}(O_\alpha)\}_\alpha$ ... it covers K and is open.
But.. K compact $\Rightarrow \exists \{f^{-1}(O_{\alpha_i})\}$ that covers K for $i \in [1,n]$. So $\{f(f^{-1}(O_{\alpha_i}))\}$ cover $f(K)$... then O_{α_i}, $i=1,\dots n$ covers $f(K)$
$\Rightarrow f$ preserves compact sets.

Th^m/ If X is Hausdorff. Then if $K \subset X$ is compact, K is closed.

__Pf__:



Want K^c open.

Pick $x_0 \in K^c$. $\forall a \in K$, $\exists N(a) \ni$

$N(a) \cap N(x_0, a) = \emptyset$.

But $\bigcup_{a \in K} N(a)$ covers K (and are open)

$\therefore \exists \{N(a_i)\}_{1 \le i \le n} \ni \bigcup_{i=1}^{n} N(a_i) \supset K$.

Look @ $N(x_0, a_i)$'s

So: $\mathcal{O}_{x_0} = \bigcap_{i=1}^{n} N(x_0, \alpha_i)$ is <u>open</u> set

and $\mathcal{O}_{x_0} \cap K = \phi$.

Now take $\bigcup_{x_0 \in K^c} \mathcal{O}_{x_0} = \mathcal{O} := K^c$

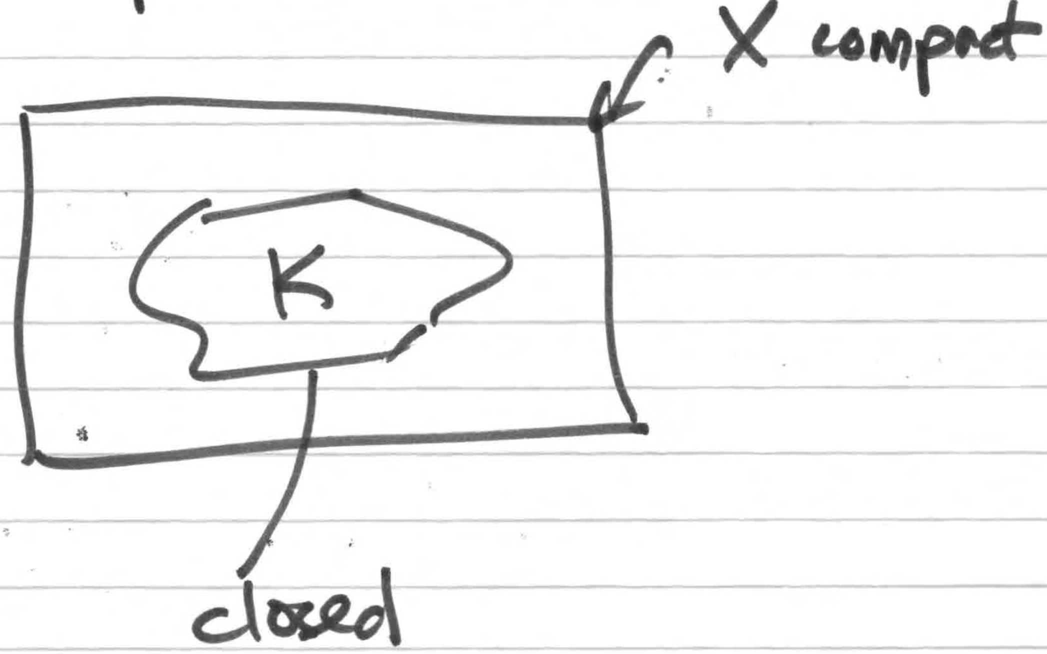
Since K^c open, K closed ∎

<u>Converse</u>: If X is compact Hausdorff space, then

if K is closed ~~and X compact~~,

K is compact.

Pf:



Let $\{O_\alpha\}_\alpha$ cover K

Note $X \setminus K = K^c$ is open

So $\{O_\alpha\}_\alpha \cup K^c$ cover X, compact

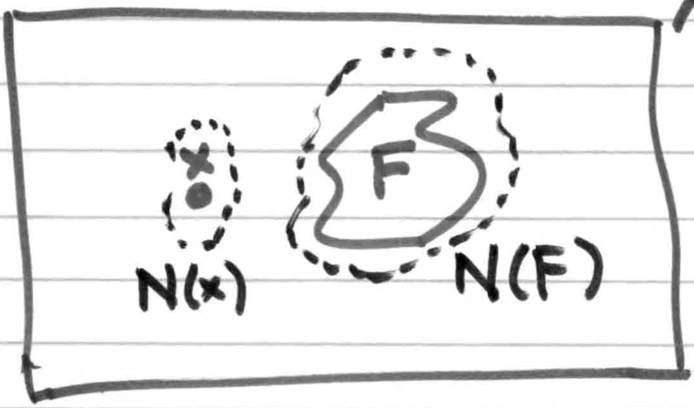
$\therefore$ $K^c \cup \{O_{\alpha_i}\}_{1 \le i \le n}$ cover X

$\Rightarrow \bigcup_{i=1}^{n} O_{\alpha_i} \supset K$. so K compact. ∎

Th^m If X is Hausdorff and X compact then X is T_3.

Pf

X is compactum



Want $N(x) \cap N(F) = \emptyset$.

Note: F compact per prior theorem.

$\forall a \in F$ construct (by Harsdorff)

$N(a)$ and $N(x,a)$ $\ni$

$N(a) \cap N(x,a) = \emptyset$.

Claim: $\bigcup_{a \in F} N(a) \supset F$, but then

since F compact, $\exists \{N(a_i)\}_{1 \le i \le n}$,

$\ni \bigcup_{i=1}^{n} N(a_i) \supset F$

Define $\mathcal{O}_x := \bigcap_{i=1}^{n} N(x, a_i)$

so $U = \bigcup_{i=1}^{n} N(a_i) \supset F$

$V = \mathcal{O}_x$

with $U \cap V = \emptyset$. ∎