Urysohn's Lemma:

Suppose X is top. sp. & F_1, F_2 are closed sets in X. Suppose X is T_4. Also $F_1 \cap F_2 = \phi$.

F_1, cl F_2, cl

Then $\exists f : X \to [0,1]$ which is continuous and $f(F_1) = 0$ & $f(F_2) = 1$.

(sometimes we call X functionally normal).

Need some lemmas:

Lemma 1: If X is T_4 and $H \supset F$ with H open and F closed, then...

there exists G, open $\ni$

$$\underset{\text{open}}{H} \supset \underset{\text{closed}}{\mathrm{cl}\,G} \supset \underset{\text{open}}{G} \supset \underset{\text{closed}}{F}.$$

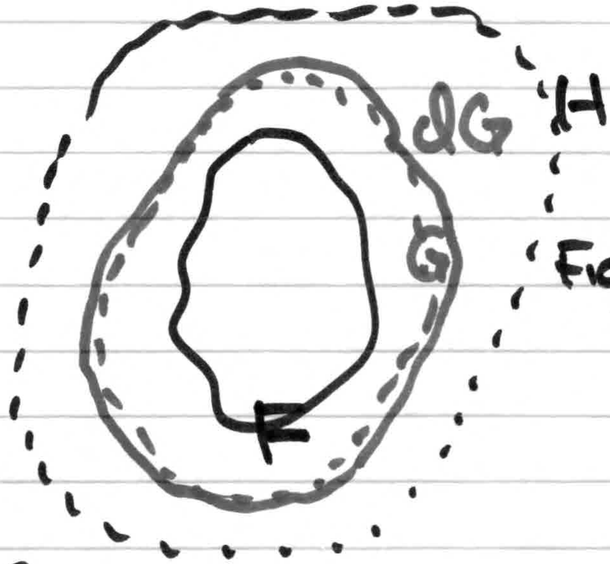


Fig. 1

nota bene

closed

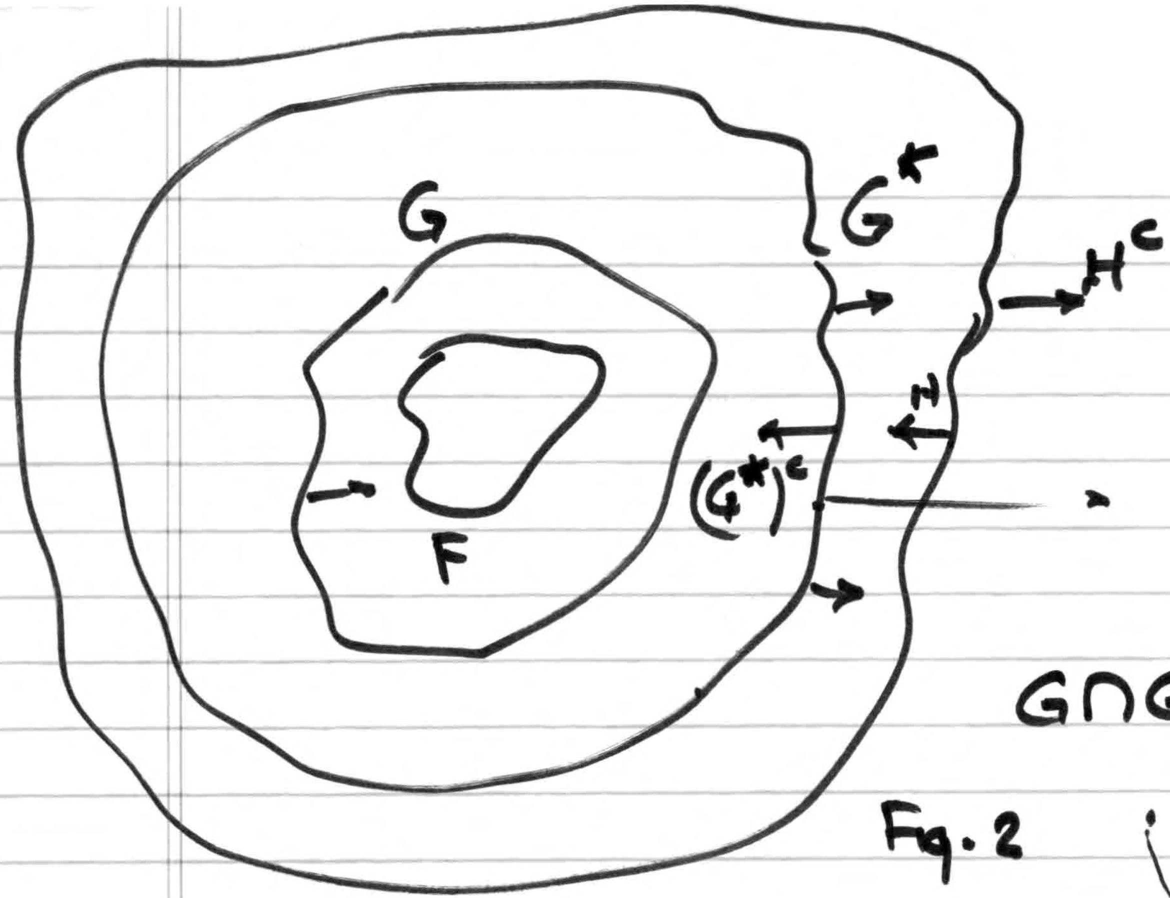
NB: H^c is closed. $F \cap H^c = \emptyset$.

By T_4 prop. $\exists$ open sets $G, G^* \ni$

$F \subset G$ and $H^c \subset G^*$. Moreover,

$G \cap G^* = \emptyset$. $\Rightarrow G \subset (G^*)^c$.

Also, since $H^c \subset G^*$ $(G^*)^c \subset H$.



$G \cap G^* = \phi$

Fig. 2

We have $F \subset G \subset G^{*c} \subset H$

closed, since G^* open

Hence $F \subset G \subset \operatorname{cl} G \subset H$ as req'd. ■

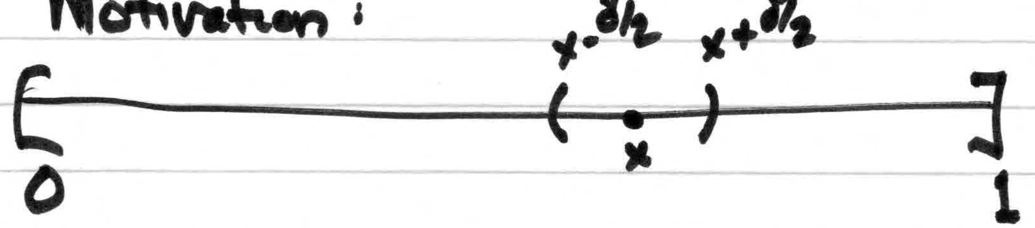
Lemma 2: Let D be set of dyadic fractions in $[0, 1]$.

$$D = \left\{ \frac{k}{2^n} : k, n \in \mathbb{N}, k \leq 2^n \right\}$$

p-adic

$\mathrm{cl}\, D = [0,1]$.

Pf: Motivation:



Given $x \in [0,1]$ and $N(x,\delta)$ need to find element of D in $\bar{N}(x,\delta)$.

By hashmark argument $\exists\, n \ni k/2^n \in (x-\delta/2, x+\delta/2)$, hence $x \in D'$

so $\mathrm{cl}\, D \ni x$, hence $[0,1] = \mathrm{cl}\, D$.

Urysohn's Lemma:

Given X is T_4 and F_1, F_2 closed subsets of X, $\exists\, f : X \to [0,1] \ni f$ is cont. and $f(F_1) = 0$ and $f(F_2) = 1$.

i.e. X is "functionally normal".

Given $F_1 \cap F_2 = \emptyset$. So F_2^c open and

$\therefore \underset{\text{open}}{F_2^c} \supset \underset{\text{closed}}{F_1}$.

$$\overset{\text{open}}{F_2^c} \supset \overset{\text{cl}}{\mathrm{cl}\, G_{1/2}} \supset \overset{\text{op}}{G_{1/2}} \supset \overset{\text{cl}}{F_1}$$

<u>Iterate:</u>

$$F_2^c \supset \mathrm{cl}\, G_{3/4} \supset G_{3/4} \supset \mathrm{cl}\, G_{1/2} \supset G_{1/2} \supset \mathrm{cl}\, G_{1/4} \supset G_{1/4} \supset F_1$$

So by recursion, we have:

for $t \in D$ $\exists \underset{\text{open}}{G_t}$ s.t. $t_1, t_2 \in D$,

if $t_1 < t_2$ $\mathrm{cl}\, G_{t_1} \subset G_{t_2}$

Define $f : X \to [0, 1]$

$$f(x) = \begin{cases} \inf \{t : x \in G_t\} & \text{if } x \in F_2 \\ 1 & \text{if } x \in F_2 \end{cases}$$