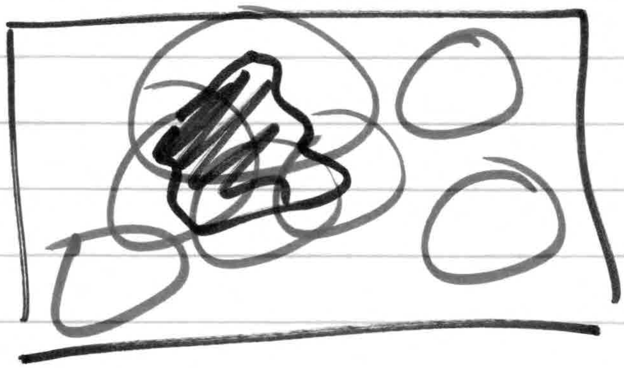
①

Convexité, Continuité, Compacité

1) A covering of $S \subset X$, (X, τ) is top. sp. is collection of sets $\{O_\alpha \in \tau\}$ $\ni$ $S \subset \bigcup_\alpha O_\alpha$ $\{O_\alpha\}$ is cover for S



open cover - O_α are open

arbitrary cover - just sets

If S is covered by open sets O_α, then if finitely many O_α's cover S

2) S is compact. [Borel-Lebesgue].

3) S is sequentially compact whenever every sequence in S has a non-stationary convergent subsequence.

$$a_n = \begin{cases} 1/n & \text{if } n \text{ even} \\ -3 & \text{if } n \text{ odd} \end{cases}$$

4) S is Bolzano-Weierstrass compact if every infinite subset of S has a limit pt.

5) S is countably compact if every countable open cover is reducible to a finite subcover.

6) S is Lindelöf if every open cover is reducible to a countable cover.

Heine-Borel Th^m/

In $\mathbb{R}$ a set is compact iff it is closed & bounded.

Digression:

Cantor Intersection Th^m.

Let C_i be a countable collection of nested closed sets in (X, ρ)

$\lim_{n \to \infty} \operatorname{diam}(C_n) = 0.$

Then $\bigcap_{n=1}^{\infty} C_n$ consists of one point

$\exists\, x \in C_n \ \forall n \in \mathbb{N}$, then consider

$y \neq x$ know $\bigcap_{n=1}^{\infty} C_n \supset \{x\}$

Suppose FSOC $\bigcap_{n=1}^{\infty} C_n \supset \{x, y\}$

$\rho(x,y) = 0$ iff $x = y$

$\rho(x,y) = \varepsilon$

So choose $\operatorname{diam}(C_n) < \varepsilon$ so

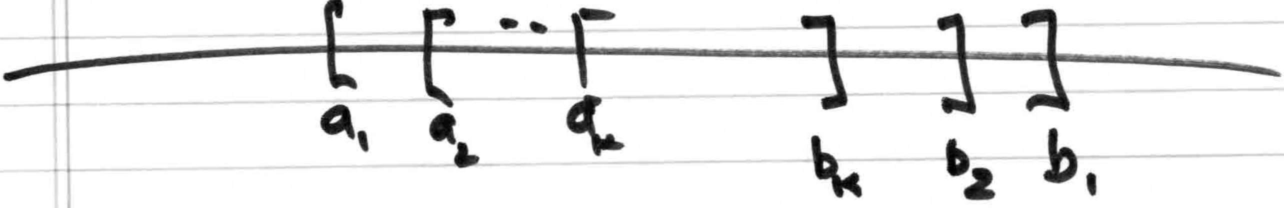
$\rho(x,y) \geq \operatorname{diam}(C_n)$ so $y \in C_n$

i.e. $\bigcap_{n=1}^{\infty} C_n \not\ni y$.

Closed Interval Thm (Cantor's $\cap$ Thm for closed intervals)

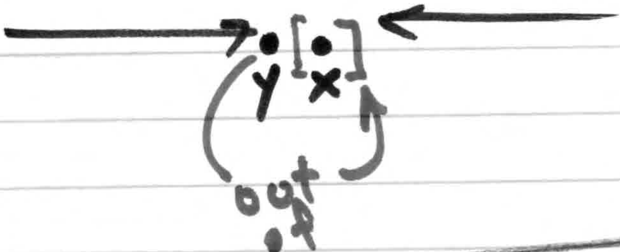
Let $[a_n, b_n]$ be nested seq. of closed intervals in $\mathbb{R}$ $\ni$ $\lim_{n\to\infty} (b_n - a_n) = 0$.

$$\Rightarrow x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$$



$a_n \uparrow \quad b_n \downarrow \qquad a_i < b_j \quad \forall i, j \in \mathbb{N}$

Consider $\sup_n a_n = x$



Proof Lite

Heine-Borel:

In $\mathbb{R}$, $K \subset \mathbb{R}$ is compact iff K is bounded & closed.