3.1 METRIC TOPOLOGIES

Maurice Fréchet coined the term “metric space” in 1906 and initiated a formal study of their properties. In a nutshell, they are topological spaces equipped with yardsticks. Specifically, the open sets in a metric space topology are formulated in terms of a distance function. This feature gives metric spaces a recognizable connection to geometric reality, and some texts present them first as a concession to the more abstract nature of general topological spaces. Metric spaces give us a hybrid setting in which to continue developing the ideas of Chapter 1, and to begin to discuss so-called uniform notions that are not expressible in purely topological terms. Metric spaces are the prototypical uniform spaces, and an abstraction of their properties led to the development of a general theory of uniformities. We will go down the uniform road only as far as metric spaces will allow, but there is much interesting territory to cover.

Definition ***3.1.1*** *Let X be a set and $\rho : X \times X \to \mathbb{R}_{0+}$ a function. If ρ satisfies the following axioms, then we call ρ a* ***pseudometric*** *(**gauge**, **semimetric**, **or écart**) for X:*

(**M1**) $\rho(x,x) = 0$
(**M2**) $\rho(x,y) = \rho(y,x)$
(**M3**) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$
The pair (X,ρ) **is called a pseudometric space**.

The terms semimetric and écart appear in the literature as inconsistent synonyms for pseudometric. The reader is advised, as usual, to check the definitional conventions in force.

Definition ***3.1.2*** *We call a pseudometric a* ***metric*** *(**distance function**) if it also satisfies:*

(**M4**) $\rho(x,y) = 0$ **implies** $x = y$
In this case, **the pair** (X,ρ) **is called a metric space**.

Context permitting, we will abbreviate our notation and refer to *X* as the metric (or pseudometric) space.

The axioms in §3.1.1 and §3.1.2 are sometimes described informally by saying that a metric displays reflexivity (M1), symmetry (M2), satisfies the triangle inequality (M3), and is faithful (M4). The term *semimetric* (obviously not in the sense of §3.1.1) is also occasionally used to describe a metric for which M2 is relaxed. Symmetry can always be recovered by taking such a semimetric ρ and defining $\hat{\rho}(x,y) = \rho(x,y) + \rho(y,x)$. The function $\hat{\rho}$ is then a proper metric.

One salient drawback of pseudometric spaces is that they are not Hausdorff or even weakly separated, since pseudometrics cannot tell the difference between points that are separated by zero distance. Many important results for metric spaces nevertheless carry through for pseudometric spaces. We will focus most of our attention on metrics, but with sufficient enthusiasm for mutatis mutandi arguments, the reader may create a roughly parallel body of results for pseudometrics. [Kel55] develops everything in terms of pseudometrics and flags those results where M4 is required. What a single pseudometric cannot do, a collection of them sometimes can.

Definition ***3.1.3*** *Let $\{\rho_\alpha : \alpha \in A\}$ be a family of pseudometrics for a set X. If for every pair of points $x,y \in X$, there exists some $\beta \in A$ such that $\rho_\beta(x) \neq \rho_\beta(y)$, then we call $\{\rho_\alpha : \alpha \in A\}$ a* ***separating family*** *of pseudometrics.*

In view of M4, every metric is a “separating family” all by itself. In contrast to a single pseudometric, a separating family of them allows the Hausdorff property to be recovered.

Definition ***3.1.4*** *Let (X,ρ) be a metric space. For each $x \in X$, the set $B(x,\varepsilon) = \{y \in X : 0 \leq \rho(x,y) < \varepsilon\}$ is called an **open ball** (**sphere**) of radius ε centered at x (ε-**ball**). Correspondingly, the set $\bar{B}(x,\varepsilon) = \{y \in X : 0 \leq \rho(x,y) \leq \varepsilon\}$ is called a **closed ball** of radius ε centered at x. If the metric needs to be emphasized, we can write $B_\rho(x,\varepsilon)$.*

Proposition ***3.1.5*** *Let (X,ρ) be a metric space. Then the family $\mathfrak{B}_x = \{B(x,\varepsilon) : x \in X, \varepsilon > 0\}$ is a base for a topology on X.*

Proof We need to show that the intersection of any two members of the family contains a third member. If $z \in B(x,\varepsilon) \cap B(y,\eta)$, then consider $d_x = \rho(z,x) < \varepsilon$ and $d_y = \rho(z,y) < \eta$. We can clearly choose $d > 0$ such that $d < \min(\varepsilon - d_x, \eta - d_y)$. Apparently $B(z,d) \subset B(x,\varepsilon)$, since if $w \in B(z,d)$, then $\rho(x,w) \leq \rho(x,z) + \rho(z,w) \leq d_x + d < \varepsilon$. Similarly we have $\rho(y,w) < \eta$, and $B(z,d) \subset B(x,\varepsilon) \cap B(y,\eta)$. By §2.1.30, the result follows. ■

Definition ***3.1.6*** *The topology $\mathcal{T}$ induced by the family in §3.1.5 is called the **metric topology** on X generated by ρ. Unless noted to the contrary, we will always assume a metric space is endowed with its metric topology.*

The following subtlety is hidden in §3.1.4. Although $B(x,\varepsilon)$ is a basic open set for the metric topology, it does not follow that $clB(x,\varepsilon) = \bar{B}(x,\varepsilon)$. Consider the discrete topology induced by the metric of §3.1.10 below. The closure of $B(x,1)$ for any $x \in X$ is just $\{x\} = B(x,1)$. However the closed ball $\bar{B}(x,1) = X$. Observe, too, that there is no uniqueness implied in the way that metrics generate topologies. Infinitely many metrics may generate the same topology.

Definition ***3.1.7*** *Let $(X,\mathcal{T})$ be a topological space. If $\mathcal{T}$ is generated by some metric ρ, then $(X,\mathcal{T})$ is said to be **metrizable**.*

Deciding what properties were required to render a topological space metrizable was one of the central problems of topology in the first half of the twentieth century. We will examine the fine points of this issue in §3.2. Occasionally, we will state theorems in terms of metrizable, rather than metric spaces. There is no real difference given the notion of equivalent metrics (see §3.1.16). The point of using “metrizable” instead of “metric” is to signal that we are considering a topological space for which we can find, if we wish, a metric that generates the given topology.

Example ***3.1.8*** *$\mathbb{R}$ with its usual topology is a metric space. The **usual metric** is absolute value: $\rho(x,y) = |x-y|$ and generates this topology. However, so do the **arctangent metric** $\rho_{\text{arctan}}(x,y) = |\arctan x - \arctan y|$ and the metric $\rho' = \dfrac{|x-y|}{1+|x-y|}$. The latter metric takes values in $[0,1]$ and hence is bounded.*

Example ***3.1.9*** *Suppose E is a normed linear space over $\mathbb{R}$, with norm $\|x\|$. Then E is a metric space under the topology generated by the metric $\rho(x,y) = \|x-y\|$. This is also called the **norm topology**. Every norm generates a metric in this simple way, but a converse construction is not available.*

Example ***3.1.10*** *Every discrete space D is metrizable by the discrete metric. For $x,y \in D$, let*

$\rho(x,y) = 0$ if $x = y$, and $\rho(x,y) = 1$ if $x \neq y$.

Example ***3.1.11*** *Let $X = l^2(\mathbb{N})$, the space of square summable sequences in $\mathbb{R}$. For $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$, define $\rho(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{n=1}^{\infty}|x_n - y_n|^2}$. Then X is a metric space with the so-called l^2-metric. (X,ρ) is known as* ***Hilbert sequence space****.*

Example ***3.1.12*** *Let X be the space of square integrable real functions on $\mathbb{R}$ with respect to ordinary Lebesgue measure. For $f,g \in X$. define $\rho(f,g) = \left(\int_R |f(\xi) - g(\xi)|^2 d\lambda\right)^{\frac{1}{2}}$. Then ρ is a pseudometric for X. If we identify functions that are equal almost everywhere, and call this equivalence relation R, then X/R is a metric space for ρ, which is called the L^2-metric (as well as norm).*

Definition ***3.1.13*** *Let (X,ρ) be a metric space. If $\sup\{\rho(x,y) : x,y \in X\} < \infty$, we say that ρ is a* ***bounded metric****.*

Example ***3.1.14*** *Let $X = \mathbb{N}$, and define $\rho(m,n) = |\frac{1}{m} - \frac{1}{n}|$. This metric is bounded by 1 and induces the discrete topology on the positive integers.*

Every metric has associated with it a **standard bounded metric**. Both the original metric and the associated bounded metric generate the same topology. This is a generalization of the bounded metric for the reals which generates the usual topology on the reals in §3.1.8.

Proposition ***3.1.15*** *Let (X,ρ) be a metric space with metric topology $\mathcal{T}$. Then there exists a bounded metric $\hat{\rho}$ taking values in $[0,1]$ such that the topology induced by $\hat{\rho}$ is $\mathcal{T}$.*

Proof Define $\hat{\rho}(x,y) = \dfrac{\rho(x,y)}{1+\rho(x,y)}$. Evidently $\hat{\rho} : X \times X \to [0,1]$. The axioms M1, M2, and M4 are obviously satisfied. Only M3 requires some elaboration. Consider $f(t) = \dfrac{t}{1+t}$ for $t \in [0,\infty)$. Evidently $f'(t) > 0$ on $[0,\infty)$, so $f(t)$ is increasing. Hence if $a+b \geq c$, where $a,b,c \in [0,\infty)$, we must have $f(a+b) \geq f(c)$, or $\dfrac{a+b}{1+a+b} \geq \dfrac{c}{1+c}$. Also, $\dfrac{a}{1+a} \geq \dfrac{a}{1+a+b}$ and $\dfrac{b}{1+b} \geq \dfrac{b}{1+a+b}$, so combining results we get $\dfrac{a}{1+a} + \dfrac{b}{1+b} \geq \dfrac{c}{1+c}$. Since ρ satisfies the triangle inequality, we can set $a = \rho(x,y)$, $b = \rho(y,z)$, and $c = \rho(x,z)$. But this is equivalent to$\hat{\rho}(x,y) + \hat{\rho}(y,z) \geq \hat{\rho}(x,z)$, which shows that $\hat{\rho}$ itself satisfies the triangle inequality. Now suppose that $B_\rho(x,\varepsilon)$ is an arbitrary basic open set centered at x. If $z \in B_{\hat{\rho}}(x,\eta)$, then $\hat{\rho}(x,z) < \eta$ implies $\dfrac{\rho(x,y)}{1+\rho(x,y)} < \eta$. This in turn occurs whenever $\rho(x,z) < \dfrac{\eta}{1-\eta}$. So if we choose $\dfrac{\eta}{1-\eta} < \varepsilon$, or equivalently $\eta < \dfrac{\varepsilon}{1+\varepsilon}$, then $z \in B_p(x,\varepsilon)$. Hence for $\eta < \dfrac{\varepsilon}{1+\varepsilon}$, $B_{\hat{\rho}}(x,\eta) \subset B_p(x,\varepsilon)$. A similar argument shows that for suitable δ, depending on η, $B_p(x,\delta) \subset B_{\hat{\rho}}(x,\eta)$. Thus the base for the metric topology corresponding to $\hat{\rho}$ is contained in the abse for the metric topology corresponding to ρ, and vice versa. It follows that the two metric topologies are identical. ■

A different metric satisfying the conditions of the preceding proposition is $\hat{\rho}(x,y) = \min(1,\rho(x,y))$, which we have as an optional standard bounded metric. This metric agrees with the given one “in the small”, and so agrees topologically. For either associated bounded metric,

id_X is a homeomorphism between (X,ρ) and $(X,\hat{\rho})$, so for topological purposes, we need only consider bounded metric spaces.

Definition ***3.1.16*** *If ρ_1 and ρ_2 are two metrics which generate the same topology on a set X, then ρ_1 and ρ_2 are said to be* ***equivalent*** *and this situation is denoted by $\rho_1 \sim \rho_2$. As might be guessed, "$\sim$" is an equivalence relation on the family of all metrics for a fixed set.*

Example ***3.1.17*** *A metric ρ and its standard bounded associate $\hat{\rho}$ are equivalent metrics by §3.1.15.*

Example ***3.1.18*** *The* ***euclidean metric*** *on $\mathbb{R}^n$ is defined to be $\rho_e(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, where $\mathbf{x} \in \mathbb{R}^n$ is the vector $(x_1,\ldots x_n)$. $\mathbb{R}^n$ with the corresponding metric topology is customarily written E^n. $\mathbb{R}^n$ with the usual topology is exactly E^n, however, we can also get the usual topology with the metrics $\rho_{\max}(\mathbf{x},\mathbf{y}) = \max_{1\le i\le n}|x_i - y_i|$ and $\rho_{\square}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n}|x_i - y_i|$. We reserve the notation E^n for the metric space $(\mathbb{R}^n,\rho_e)$, even though equivalent metrics, such as $\rho_{\max}$ and $\rho_{\square}$ generate the same topology. The metric $\rho_{\square}$ is called the* ***square metric****. Applied to $\mathbb{R}^2$ or $\mathbb{Z}^2$, it is sometimes fancifully referred to as the* ***taxicab metric****. The analogy comes from associating each dimension with a set of orthogonal "streets". Distances are measured by driving as a taxi would along these streets, instead of by using the Pythagorean formula.*

Definition ***3.1.19*** *Let (X,ρ) be a metric space and $Y \subset X$. The* ***diameter*** *of Y is the number $diamY = \sup\{\rho(x,y) : x,y \in Y\}$. We define $diam\varnothing = 0$.*

Definition ***3.1.20*** *Let (X,ρ) be a metric space and $Y \subset X$. If there exists a $\beta \in \mathbb{R}$ such that $diamY \le \beta$, then we call Y a* ***bounded set****.*

Another way of saying §3.1.13 is to declare that a *metric* is bounded if the *whole space* is bounded.

Definition ***3.1.21*** *Let (X,ρ) be a metric space and $Y,Z \subset X$. The* ***distance*** *from Y to Z is the number $dist(Y,Z) = \inf\{\rho(y,z) : y \in Y, z \in Z\}$. If either set is a singleton, we suppress the curly bracket notation, and write, for example, $dist(y,Z)$. If the metric needs to be emphasized, we will write $dist_\rho(Y,Z)$.*

If $dist(Y,Z) > 0$, then clearly $Y \cap Z = \varnothing$. The converse fails, even if the two sets are closed. Consider $\mathbb{R}$ with its usual metric and the two subsets $Y = \{n : 1 < n \in \mathbb{N}\}$ and $Z = \{n + \frac{1}{n} : 1 < n \in \mathbb{N}\}$. These are infinite disjoint closed sets, yet $dist(Y,Z) = 0$.

Proposition ***3.1.22*** *Let (X,ρ) be a metric space and $Y \subset X$. Then $dist(x,Y) = 0$ if and only if $x \in clY$.*

Proof $x \in clY$ if and only if for every $\varepsilon > 0$, the basic set $B(x,\varepsilon)$ meets Y. In turn, this occurs if and only if there is a $y \in Y$ such that $\rho(x,y) < \varepsilon$,for every $\varepsilon > 0$. But this latter statement is true if and only if $dist(x,Y) = \inf\{\rho(x,y) : y \in Y\} = 0$. ■

Proposition ***3.1.23*** *Let ρ and σ be metrics on X. Then $\rho \sim \sigma$ if and only if for every $Y \subset X$, $dist_\rho(x,Y) = 0$ implies $dist_\sigma(x,Y) = 0$, and conversely.*

Proof (Necessity) Suppose $dist_\rho(x,Y) = 0$ precisely whenever $dist_\sigma(x,Y) = 0$. By §3.1.22, x belongs to the $\mathcal{T}_\rho$-closure of Y if and only if it belongs to the $\mathcal{T}_\sigma$-closure of Y. But Y is arbitrary, so the two closure operations are identical on X. Hence the two metric topologies must agree and $\rho \sim \sigma$.

(Sufficiency) If ρ and σ are equivalent metrics, then they generate the same topology, hence the same closed sets. Then $dist_\rho(x,Y) = 0$ is equivalent to $x \in clY$, which is in turn equivalent to $dist_\sigma(x,Y) = 0$ by §3.1.22. ■

The notion of continuity can be stated for metric spaces in the comfortable terms of epsilontics, by which some authors mean the familiar ε-δ machinery of calculus. In this setting, the concept of homeomorphism has a stronger cousin, which is not only a bicontinuous bijection, but a distance-preserving map as well. The class of metric space invariants is determined by the effect of these more impeccably behaved maps, and will therefore differ from the class of purely topological invariants for a given space.

Definition ***3.1.24*** *Let (X,ρ) and (Y,σ) be metric spaces and $f : X \to Y$. Given $x_0 \in X$, if for every $B_\sigma(f(x_0),\varepsilon)$ there exists some $B_\rho(x_0,\delta)$ such that $f(B_\rho(x_0,\delta)) \subset B(f(x_0),\varepsilon)$, then we say f is* ***continuous at*** *x_0. An equivalent statement is that given any $\varepsilon > 0$, if there exists a $\delta > 0$ such that $\rho(x,x_0) < \delta$ implies $\sigma(f(x),f(x_0)) < \varepsilon$, then f is continuous at x_0. If f is continuous at x for all $x \in Y \subset X$, then we say f is* ***continuous on*** *Y.*

The preceding definition is a restatement of §2.5.2 using the metric open sets available. For $\mathbb{R}$ $(= E)$, this is the familiar requirement that given x_0, then for any preassigned $\varepsilon > 0$, we must be able to find a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

Definition ***3.1.25*** *Let (X,ρ) and (Y,σ) be metric spaces. If $d : X \to Y$ is a function with the property that for all $x_1,x_2 \in X$ we have $\rho(x_1,x_2) = \sigma(d(x_1),d(x_2))$, then we call d a* ***distance-preserving map****.*

Definition ***3.1.26*** *Let (X,ρ) and (Y,σ) be metric spaces and $\psi : X \to Y$ a distance-preserving map. If ψ is a surjection, then we say it is an* ***isometry****, and the spaces (X,ρ) and (Y,σ) are* ***isometric****, denoted by $(X,\rho) \overset{iso}{\cong} (Y,\sigma)$, or $X \overset{iso}{\cong} Y$.*

Example ***3.1.27*** *Every translation of the plane E^2 is given by a map of the form $(x,y) \mapsto (x+a,y+b)$. This corresponds to a rigid translation of the plane with no rotation, so that the origin is moved to the point (a,b). It is easy to see from the form of the euclidean metric that every map of this type is an isometry. Taken together, these maps form a group which is a subgroup of the* ***group of isometries of the plane****. The group of isometries admits rotations and reflections in addition to translations.*

Proposition ***3.1.28*** *Let (X,ρ) and (Y,σ) be metric spaces, and $d : X \to Y$ a distance-preserving function. Then d is a homeomorphism between X and $d(X) \subset Y$.*

Proof Note that if $x_1 \neq x_2$ in X, then $d(x_1) \neq d(x_2)$ in Y, hence d is injective. Now if $B(y,\varepsilon)$ is an open ball in Y at $y = d(x)$, then $B(x,\varepsilon)$ is an open ball in X at x such that $d(B(x,\varepsilon)) \subset B(y,\varepsilon)$, by the distance preservation property of d. Clearly this argument can be reversed, and we conclude that d is bicontinuous. Thus d is a bicontinuous bijection from X to $d(X)$, and the result is immediate. ■

Note that the preceding proposition would not be true for a pseudometric, which would be insufficient to establish injectivity.

Corollary *3.1.29* *Every isometry between metric spaces is a homeomorphism.*

Not every homeomorphism between metric spaces is an isometry, however. To distinguish the two levels of isomorphism, truly isomorphic spaces are sometimes referred to as being **isometrically isomorphic**. The next example presents two spaces which are isomorphic as topological spaces, but not as metric spaces.

Example *3.1.30* *Consider $\mathbb{N}$ with the two discrete metrics $\rho_i(m,n) = i$ if $m \neq n$ and $\rho_i(m,n) = 0$ if $m = n$ for $i = 1,2$. Clearly $id_{\mathbb{N}}$ is a homeomorphism between the discrete spaces, but it is trivially not an isometry.*

Definition *3.1.31* *Let (X,ρ) be a fixed metric space, and (Y,σ) be any metric space that is isometrically isomorphic to (X,ρ). If (Y,σ) possesses property $\mathcal{P}$ whenever (X,ρ) does, then property $\mathcal{P}$ is called a **metric invariant**.*

Example *3.1.32* *Any topological invariant is going to be a fortiori a metric invariant. We know, for example, that the boundedness of a set is not a topological property, but the additional constraint of distance preservation makes it a metric invariant.*

One of those nearly trivial, but occasionally useful, observations is that a metric itself is continuous for the product topology formed from the associated metric topology on each factor. The following proposition shows that distance between a point and a fixed set in a metric space is also a continuous function.

Proposition *3.1.33* *Let (X,ρ) be a fixed metric space and $Y \subset X$. Then the map $d : X \to [0,\infty)$; $x \mapsto dist(x,Y)$ is continuous.*

Proof For any $x,z \in X$, and any $y \in Y$, we have $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. Then $dist(x,Y) = \inf\{\rho(x,y) : y \in Y\} \leq \rho(x,z) + \inf\{\rho(z,y) : y \in Y\} \leq \rho(x,z) + dist(z,Y)$. Now $d(x) \leq \rho(x,z) + d(z)$, or by symmetry, $|d(x) - d(z)| \leq \rho(x,z)$. It follows that if $\rho(x,z)$ is sufficiently small, then $|d(x) - d(z)|$ can be made smaller than any preassigned $\varepsilon > 0$. Hence the distance function d is continuous. ■

Let us now turn to a discussion of the topological properties of metric spaces, including their stability under various constructions and mappings.

Proposition *3.1.34* *Let (X,ρ) be a metric space. Then (X,ρ) is first countable.*

Proof Suppose $\mathcal{B}_x = \{B(x,\varepsilon) : \varepsilon > 0\}$ is a base at x for the metric topology. By the archimedean property of the reals, for every $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Hence $\mathcal{B}'_x = \{B(x,\frac{1}{n}) : n \in \mathbb{N}\}$ is also a base for the metric topology, and since this base is countable, the result is clear. ■

Definition *3.1.35* *The base $\mathcal{B}'_x$ subordinate to $\mathcal{B}_x$ in the preceding proposition will be referred to as the **standard (canonical) countable base** for the metric topology at x.*

Definition *3.1.36* *Let (X,ρ) be a metric space and $Y \subset X$. Y is said to be ε-**dense** in X if $\bigcup\{B(y,\varepsilon) : y \in Y\} \supset X$, or equivalently, if $dist(x,Y) < \varepsilon$ for all $x \in X$.*

Definition *3.1.37 Let (X, ρ) be a metric space. A set S is an ε-**net** for X if S is finite and ε-dense in X.*

Proposition *3.1.38 Let (X, ρ) be a metric space. Then the following are equivalent:*

(i) (X, ρ) **is separable**
(ii) (X, ρ) **is second countable**
(iii) (X, ρ) **is Lindelöf**

Proof $(i) \Rightarrow (ii)$ Suppose X is separable, then there exists a countable set S such that $clS = X$. If U is any open set, then there exists some $s \in S$ with $s \in U$, and hence a canonical basic open set $B(s, \frac{1}{n}) \subset U$. Now $\{B(s, \frac{1}{n}) : s \in S, n \in \mathbb{N}\}$ is a countable family of open sets, and since U was arbitrary, evidently X is second countable.

$(ii) \Rightarrow (iii)$ Any open cover $\{U_\alpha : \alpha \in A, \text{arbitrary}\}$ of X has a refinement by members of a countable global base $\{B_n : n \in \mathbb{N}\}$. For each n, we may find a suitable $U_\alpha \supset B_n$, which we relabel as U_n. Then $\{U_n : n \in \mathbb{N}\}$ is a countable subcover and hence X is Lindelöf.

$(iii) \Rightarrow (i)$ Fix n and cover X with the family $\{B(x, \frac{1}{n}) : x \in X\}$. By the Lindelöf property, there is a countable subcollection $\{B(x_i, \frac{1}{n}) : x_i \in X, i \in \mathbb{N}\}$ which covers X. Let $C_n = \{x_i : i \in \mathbb{N}\}$ be the corresponding set of centers for the countable cover of radius $\frac{1}{n}$. Claim: $C = \bigcup_{n \in \mathbb{N}} C_n$ is a countable dense subset of X. Each C_n is countable, so surely C is countable as well. Given any open set U in X, there is certainly a basic set $B(x, \frac{1}{n}) \subset U$ for some x and n. This set need not be in the countable subfamily chosen, hence x need not be in C. However, some member of $\{B(x_i, \frac{1}{n}) : x_i \in X, i \in \mathbb{N}\}$ must contain x, and it follows that $x_i \in B(x, \frac{1}{n})$, since x and x_i are no further apart than $\frac{1}{n}$. Accordingly, $U \cap C \neq \emptyset$, and since U was arbitrary, the claim is true. The result follows immediately. ■

If a metric space and a topological space without a metric are homeomorphic, then the metric can be exported to the topological space, and the the homeomorphism is converted to an isometry.

Proposition *3.1.39 Let (X, ρ) be a metric space, $(Y, \mathcal{S})$ a topological space, and $\Phi : X \to Y$ a homeomorphism. Then $(Y, \mathcal{S})$ is metrizable.*

Proof Define $\sigma : Y \times Y \to [0, \infty)$ by $\sigma(y_1, y_2) = \rho(\Phi^{-1}(y_1), \Phi^{-1}(y_2))$. This function is well-defined on all of Y by the bijectivity of Φ, and it is clearly a metric due to the properties of ρ, establishing the result. ■

Definition *3.1.40 Let (X, ρ) be a metric space and $Y \subset X$. The restriction of ρ to $Y \times Y$ retains the properties of a metric, and we call $(Y, \rho|_{Y \times Y})$ a **metric subspace** of (X, ρ).*

We need to establish that relativizing the open sets of a metric topology to a subspace gives the same result as first restricting the metric as in §3.1.40, and then forming open sets with this metric in the subspace.

Proposition *3.1.41 Let (X, ρ) be a metric space and $(Y, \rho|_{Y \times Y})$ a metric subspace of (X, ρ). Then the metric topology of the subspace $(Y, \rho|_{Y \times Y})$ is the same as the metric topology on X relativized to Y.*

Proof Denote the relative metric $\rho|_{Y\times Y}$ by ρ'. First, suppose that U is open in Y. For each $y \in U$, we can find some $B_{\rho'}(y,\varepsilon_y)$ such that $B_{\rho'}(y,\varepsilon_y) \subset U$. But $B_{\rho'}(y,\varepsilon_y) = B_\rho(y,\varepsilon_y) \cap Y$ by the definition of ρ'. Hence $U = \bigcup_{y\in Y} B_{\rho'}(y,\varepsilon_y) = \bigcup_{y\in Y}(B_\rho(y,\varepsilon_y) \cap Y) = \left(\bigcup_{y\in Y} B_\rho(y,\varepsilon_y)\right) \cap Y = G \cap Y$, where G is open in X. Thus every set open in the metric topology induced by ρ' on Y is a relativization of an open set in the metric topology induced by ρ on X. Conversely, suppose V is a ρ-open set. Consider $V \cap Y$. For each $x \in V$, there exists a $B_\rho(x,\varepsilon_x)$ such that $B_\rho(x,\varepsilon_x) \subset V$. Evidently, $V \cap Y = (\bigcup_{x\in V} B_\rho(x,\varepsilon_x)) \cap Y = \bigcup_{x\in V}(B_\rho(x,\varepsilon_x) \cap Y) = \bigcup_{x\in V} B_{\rho'}(x,\varepsilon_x)$, and we see that $V \cap Y$ is a ρ'-open set. It follows that the ρ'-open sets are precisely the ρ-open sets relativized to Y, which is the required result. ■

We can use this proposition to show that a given topological space is metrizable by embedding it topologically into a known metric space.

Proposition ***3.1.42*** *Let (X,ρ) be a metric space. Then (X,ρ) is completely normal.*

Proof First we show X is normal. Let $F,G \subset X$ be closed and $F \cap G = \varnothing$. Define $U = \{x \in X : dist(x,F) - dist(x,G) < 0\}$ and $V = \{x \in X : dist(x,F) - dist(x,G) > 0\}$. In view of §3.1.33, the function $x \mapsto dist(x,F) - dist(x,G)$ is continuous, hence U and V, as preimages of open half-lines, are open in X. They are necessarily disjoint by construction. Evidently $F \subset U$ and $G \subset V$, and normality follows at once. By §3.1.41, each subspace of X is metrizable, hence normal by the preceding argument, and thus X is completely normal. ■

Corollary ***3.1.43*** *Let (X,ρ) be a metric space. Then (X,ρ) is T_5.*

Proof We show that X is T_1. If $x,y \in X$ and $x \neq y$, then $\rho(x,y) = \eta > 0$. Choose $\varepsilon < \eta$, then $x \in B(x,\varepsilon)$ and $y \notin B(x,\varepsilon)$, as required for the T_1 property. By §3.1.42, the result is clear. ■

From §3.1.43 and §2.7.73 we see that metric spaces satisfy all of the T_i axioms.

Metrizability is not stable under the formation of unrestricted products, however countable products always allow the construction of a product metric based on the individual metrics of the factor spaces.

Example ***3.1.44*** *The space $\mathbb{R}^{[0,1]}$ is a product of uncountably many copies of $\mathbb{R}$ .Let Y be the set of all functions which equal the constant 1 function except on a finite set of coördinates. Claim #1: The constant 0 function belongs to clY. Suppose U is a basic product open set of the form $\prod_{i\in H}(-\delta,\delta) \times \prod_{\alpha\in[0,1]-H} \mathbb{R}_\alpha$, where $H \subset [0,1]$ is a finite set of reals, $\mathbb{R}_\alpha = \mathbb{R}$ for all $\alpha \in [0,1]$, and $\delta_i > 0$ for all $i \in H$. Evidently $0 \in U$. But the function which is 1 on coördinates in $[0,1] - H$ and 0 on coördinates in H certainly belongs to Y, hence $U \cap Y \neq \varnothing$. Since U was arbitrary, claim #1 is valid. Claim #2: No sequence in Y converges to 0. Suppose to the contrary that the sequence (y_n) converges to 0. Let H_n be the finite set of coördinates upon which y_n differs from 1. $\bigcup_{n\in\mathbb{N}} H_n$ is then a countable set of coördinates, which clearly cannot not equal $[0,1]$. Pick $\beta \in [0,1] - \bigcup_{n\in\mathbb{N}} H_n$ and consider the product open set $V = U_\beta \times \prod_{\alpha\in[0,1]-\{\beta\}} \mathbb{R}_\alpha$, where U_β contains 0 and excludes 1. Apparently $0 \in V$, but no member of the sequence (y_n) is contained in V, contrary to the assumption of convergence. The contradiction* *establishes claim #2. Finally, we observe that if* $\mathbb{R}^{[0,1]}$ *were metrizable, then by §3.1.43, it would be* T_1*, and then by §2.8.20 some nonconstant sequence in Y would converge to* 0*. Since such convergence has been ruled out,* $\mathbb{R}^{[0,1]}$ *cannot be metrizable.*

Proposition ***3.1.45*** *Let* (X_n, ρ_n) *be a countable family of metric spaces. Then* $X = \prod X_n$ *is metrizable.*

Proof We will construct a metric for X and show that the topology it induces is the product topology. Let $\hat{\rho}_n$ be the standard bounded metric associated with ρ_n. Consider the function $\rho : X \times X \to [0, 1)$ given by $\rho(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} \hat{\rho}_n(x_n, y_n)$. We claim that this is a metric for X. Certainly $\rho(\mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}, \mathbf{x})$ and $\rho(\mathbf{x}, \mathbf{x}) = 0$ from the corresponding properties of each $\hat{\rho}_n$. Also, since the defining series is absolutely convergent, $\rho(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} \hat{\rho}_n(x_n, y_n) \leq \sum_{n=1}^{\infty} 2^{-n}(\hat{\rho}_n(x_n, z_n) + \hat{\rho}_n(z_n, y_n)) = \sum_{n=1}^{\infty} 2^{-n} \hat{\rho}_n(x_n, z_n) +$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$. We note that $\rho(\mathbf{x}, \mathbf{y}) = 0$ implies $x_n = y_n$ for all n, hence $\mathbf{x} = \mathbf{y}$. Axioms M1 through M4 are satisfied by ρ, hence we conclude it is a metric for X.

Now consider a product subbasic open set U containing $\mathbf{x}$. $U = \pi_k^{-1}(U_k)$ for some open set $U_k \subset X_k$. Since $x_k \in U_k$, we can find $\varepsilon > 0$ such that $B_{\rho_k'}(x_k, \varepsilon) \subset U_k$, where $B_{\rho_k'}(x_k, \varepsilon)$ is an open metric ε-ball in X_k. Claim: $B_\rho(\mathbf{x}, 2^{-k}\varepsilon) \subset U$. For suppose $\mathbf{y} \in B_\rho(\mathbf{x}, 2^{-k}\varepsilon)$. Then $\rho(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} \hat{\rho}_n(x_n, y_n) < 2^{-k}\varepsilon$, so in particular $2^{-k} \hat{\rho}_k(x_k, y_k) < 2^{-k}\varepsilon$, or $\hat{\rho}_k(x_k, y_k) < \varepsilon$. But this means $y_k \in U_k$, and since every other coördinate of U is the full space, $y \in U$. Our claim that the metric ball $B_\rho(\mathbf{x}, 2^{-k}\varepsilon)$ is contained in the product subbasic open set U follows immediately., and since U was arbitrary, the metric topology is a base for the product topology.

Finally, let us take an arbitrary metric ball $B_\rho(\mathbf{x}, \varepsilon)$ and show that it contains a product open set. Choose $N \in \mathbb{N}$ sufficiently large so that $2^N(\frac{\varepsilon}{2}) > 1$. This implies that $\sum_{k=N+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. Now define $V = \bigcap_{k=1}^{N} \pi_k^{-1}(B(x_k, \frac{\varepsilon}{2}))$. V is a product open set which we claim is contained in $B_\rho(\mathbf{x}, \varepsilon)$. To see this , let $\mathbf{y} \in V$. For coördinates $1 \leq k \leq N$, $\hat{\rho}_k(x_k, y_k) < \frac{\varepsilon}{2}$ by construction. But $\rho(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{\infty} 2^{-k} \hat{\rho}_k(x_k, y_k) = \sum_{k=1}^{N} 2^{-k} \hat{\rho}_k(x_k, y_k) + \sum_{k=N+1}^{\infty} 2^{-k} \hat{\rho}_k(x_k, y_k) < \sum_{k=1}^{N} 2^{-k} \frac{\varepsilon}{2} +$ Apparently $\mathbf{y} \in B_\rho(\mathbf{x}, \varepsilon)$, and since $\mathbf{y}$ was arbitrary, our claim that $V \subset B_\rho(\mathbf{x}, \varepsilon)$ is true. We have shown that the product topology is also a base for the metric topology. Combining results, it follows that the metric and product topologies are identical. ■

Here is another metric which can be defined on an arbitrarily large product, but which does not coincide with the product topology unless the number of factors is finite.

Example ***3.1.46*** *Suppose* $\{(X_\alpha, \rho_\alpha) : \alpha \in A\}$ *is a family of metric spaces for some arbitrary nonvoid index set A. Let* $X = \prod X_\alpha$ *and define* $\rho : X \times X \to [0, \infty)$ *by* $\rho(\mathbf{x}, \mathbf{y}) = \sup\{\rho_\alpha(x_\alpha, y_\alpha) : \alpha \in A\}$*. This metric treats every coördinate equally and is called the* ***uniform (supremum) metric*** *on X. If card*$(X) \geq \aleph_0$*, it is apparent that uniform metric neighborhoods are going to be more restrictive than product neighborhoods, and in this case the uniform metric topology is strictly finer than the product metric topology.*

Let us use §3.1.45 to study an important space that played a key role in early efforts to characterize those topological spaces which are metrizable.

Definition *3.1.47 The **Hilbert cube** is the set* $I^\infty = \{\xi \in \mathbb{R}^{\mathbb{N}} : \xi = (\xi_n), |\xi_n| < n^{-1}, n \in \mathbb{N}\}$.

The Hilbert cube is a subspace of Hilbert sequence space (see §3.1.11). Since the latter is a metric space under the l^2-metric, the cube is metrizable by §3.1.41.

Proposition *3.1.48 The Hilbert cube I^∞ is homeomorphic to a countable product of copies of* $[0, 1]$.

Proof Denote the metric on I^∞ induced by the l^2-metric as σ. Claim: $I^\infty \cong \prod_{n\in\mathbb{N}}[-1, 1]_n$. Consider the mapping $\Phi : I^\infty \to \prod_{n\in\mathbb{N}}[-1, 1]_n$ given by $(\xi_n) \mapsto n\xi_n$. This is evidently surjective, since any product element (α_n) has the preimage $(n^{-1}\alpha_n)$, and injective, since $\xi \neq \zeta$ implies for some n, $\xi_n \neq \zeta_n$, hence $(n\xi_n) \neq (n\zeta_n)$. It is also continuous by §2.6.3, since $\pi_n \circ \Phi$ is just multiplication by n. We need only establish that Φ^{-1} is continuous to conclude that Φ is a homeomorphism. For each n, endow $[-1, 1]_n$ with the metric $\rho_n(\xi_n, \zeta_n) = n^{-\frac{1}{4}}|\xi_n - \zeta_n|$. Here ρ_n is just the usual relative metric on $[-1, 1]$ multiplied by a scalar. Then $\prod_{n\in\mathbb{N}}[-1, 1]_n$ is metrizable by §3.1.45, and we denote the product metric by ρ. Now if $\rho(\xi, \zeta) < \varepsilon$, then $\sigma^{-1}(\Phi^{-1}(\xi), \Phi^{-1}(\zeta)) = \sqrt{\sum_{n=1}^{\infty} n^{-2}(\xi_n - \zeta_n)^2} = \sqrt{\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \left|n^{-\frac{1}{4}}(\xi_n - \zeta_n)\right|^2}$, which is dominated by $\sqrt{\sum_{n=1}^{\infty} n^{-\frac{3}{2}}} \cdot \sqrt{\sum_{n=1}^{\infty} \left|n^{-\frac{1}{4}}(\xi_n - \zeta_n)\right|^2} < \sqrt{\sum_{n=1}^{\infty} n^{-\frac{3}{2}}} \cdot \varepsilon$. Since the series $\sqrt{\sum_{n=1}^{\infty} n^{-\frac{3}{2}}}$ converges, $\sigma^{-1}(\Phi^{-1}(\xi), \Phi^{-1}(\zeta))$ can be made arbitrarily small, and the continuity of Φ^{-1} is clear. Now the map $\Psi : [-1, 1] \to [0, 1]$ given by $x \mapsto \dfrac{x+1}{2}$ is clearly a homeomorphism, and hence $\prod_{n\in\mathbb{N}}[-1, 1]_n \cong \prod_{n\in\mathbb{N}}[0, 1]_n$, which establishes the result. ■

Several facts about the Hilbert cube are now apparent. The cube is compact by Tychonoff's Theorem applied to the homeomorphic product of closed unit intervals. By §3.1.38 and its metrizability, it is both separable and second countable.

Proposition *3.1.49 As a metric subspace of $l^2(\mathbb{N})$, the Hilbert cube I^∞ has void interior.*

Proof Denote the l^2 metric by σ. For an arbitrary point $\xi \in I^\infty \subset l^2(\mathbb{N})$ and preassigned $\varepsilon > 0$, form the metric ball $B(\xi, 2\varepsilon)$. Let $\zeta^{(n)} = \xi$ except in the n^{th} coördinate, where $\zeta_n^{(n)} = \xi_n + \varepsilon$. Evidently $\sigma(\xi, \zeta^{(n)}) = \varepsilon$, uniformly for all n, hence $\{\zeta^{(n)} : n \in \mathbb{N}\} \subset B(\xi, 2\varepsilon)$. We claim, however, that $\xi \in cl(I^\infty)^c$. Clearly some point $\zeta^{(n)}$ must belong to $(I^\infty)^c$, otherwise $|\xi_n + \varepsilon| < \dfrac{1}{n}$ for every n. This is absurd unless $\varepsilon = 0$. Since ξ and ε are arbitrary, every neighborhood of every point in I^∞ contains points of the complement, and our claim is true. It follows that $(I^\infty)^c$ is dense in $l^2(\mathbb{N})$, and the Hilbert cube has no interior points. ■

Quotients of metric spaces are generally not metrizable, although there are some important special cases where this is possible. Every pseudometric space has a naturally associated metric space which is formed by identifying points that are separated by zero distance.

Proposition *3.1.50 Let (X, ρ) be a pseudometric space and $R \subset X \times X$ an equivalence relation where xRy if and only if $\rho(x, y) = 0$. Define $\tilde{\rho} : X/R \times X/R \to \mathbb{R}_{0+}$ by $\tilde{\rho}([x], [y]) = \rho(x, y)$. Then $(X/R, \tilde{\rho})$ is a metric space. Moreover, the natural map $q : X \to X/R$ is an isometry.*

Proof Note that $\tilde{\rho}$ is well-defined, since if $x_1, x_2 \in [x]$, and $y_1, y_2 \in [y]$, we have

$\rho(x_1,y_1) \le \rho(x_1,x_2) + \rho(x_2,y_2) + \rho(y_2,y_1) = \rho(x_2,y_2)$. Interchanging indices, we have $\rho(x_2,y_2) \le \rho(x_1,y_1)$, hence $\rho(x_2,y_2) = \rho(x_1,y_1)$. Conformance with axiom $M4$ has been built into the construction. $\tilde{\rho}$ obviously obeys axioms $M1$, $M2$, and $M3$ because ρ does, and hence is a metric. By definition, $\rho(x,y) = \tilde{\rho}(q(x),q(y))$, hence q is a distance-preserving map. Coupled with its surjectivity, q is evidently an isometry. ■

We conclude this section with a characterization of those topologies which are induced by separating families of pseudometrics (see §3.1.3). In this context, it is customary to call a pseudometric a **gauge** and talk in terms of separating families of gauges.

Definition ***3.1.51*** *Let X be a set, and $\mathcal{R} = \{\rho_\alpha : \alpha \in A\}$ a family of gauges which separate the points of X (in the manner of §3.1.3). Then the topology $\mathcal{T}(\mathcal{R})$ generated by the subbase $\{B_\alpha(x,\varepsilon) : x \in X, \varepsilon > 0, \alpha \in A\}$ is called the **gauge topology** induced by $\mathcal{R}$ on X. As usual, $B_\alpha(x,\varepsilon) = \{y \in X : \rho_\alpha(x,y) < \varepsilon\}$. The pair $(X,\mathcal{T}(\mathcal{R}))$ is called a **gauge space**.*

Definition ***3.1.52*** *Let $(X,\mathcal{T})$ be a topological space. If there exists a separating family of gauges $\mathcal{R}$ such that $\mathcal{T} = \mathcal{T}(\mathcal{R})$, then we say that $(X,\mathcal{T})$ admits a **gauge structure**.*

From §3.1.41 it follows that a subspace of a gauge space is a gauge space, with the subspace topology generated by the family of restrictions of the original gauges.

Proposition ***3.1.53*** *Let $(X,\mathcal{T})$ be a topological space. Then $(X,\mathcal{T})$ admits a gauge structure if and only if it is a Tychonoff space.*

Proof (Necessity) If $(X,\mathcal{T})$ is Tychonoff, then by §2.7.50, it is homeomorphic to a subspace of a cube, say $[0,1]^S$ for some suitable set S. A map of the form $\rho_s(\mathbf{x},\mathbf{y}) = |\pi_s(\mathbf{x}) - \pi_s(\mathbf{y})|$ is clearly a gauge for X, since is is a metric in the coördinate $s \in S$. The family of gauges $\mathcal{R} = \{\rho_s : s \in S\}$ separates points and generates the product topology on $[0,1]^S$, which therefore admits the gauge structure $\mathcal{R}$. The remark above indicates that any subspace of $[0,1]^S$ is a gauge space which admits $\mathcal{R}$. By the topological invariance of pseudometrizability, $(X,\mathcal{T})$ must admit the gauge structure $\mathcal{R}$.

(Sufficiency) Suppose $(X,\mathcal{T})$ is a gauge space which admits the gauge structure $\mathcal{R}$. If $x \neq y$ in X, then by the separating property of $\mathcal{R}$, there exists a gauge $\rho \in \mathcal{R}$ such that $\rho(x,y) = 2\eta$ for some $\eta > 0$. Clearly $B_\rho(x,\eta)$ is an open set in X which contains x and excludes y. Thus $(X,\mathcal{T})$ is a T_1 space. Now we show that $(X,\mathcal{T})$ is completely regular. Suppose $x \notin F$, closed in X. Then $x \in F^c$, which is open in X. Then for some $\rho \in \mathcal{R}$ and $\varepsilon > 0$, we have $B_\rho(x,\varepsilon) \subset F^c$. Let $\sigma(y) = \min\{1, \frac{\rho(x,y)}{\varepsilon}\}$. Evidently $\sigma(x) = 0$, $\sigma(F) = 1$, and σ is continuous on X But x and F were arbitrary, so this establishes complete regularity. Coupled with the T_1 property, we conclude that $(X,\mathcal{T})$ is Tychonoff. ■