2.9 COMPACTNESS & RELATED PROPERTIES

Compactness is an important topological property which can be expressed in different ways that are not obviously equivalent. We will adopt a definition that is not the most intuitively appealing, but which lends itself to proving things efficiently about compact sets. Ultimately, of course, the choice of a primitive definition is really a matter of taste. Originally, compactness was formulated as a condition that guaranteed the convergence of some subsequence of a given sequence. Today this notion is called sequential compactness, to distinguish it from the modern formulation of compactness, which has proved to be a more fruitful concept, and which has since inherited the original name. The strength of the modern idea stems from its applicability to topological spaces which are too large or too complicated to be adequately investigated by methods grounded in countability, including, of course, those depending on sequential convergence or sequential continuity.

Complicating the issue slightly is the fact that there are additional variations on the compactness theme. We will sketch the interconnections among these collateral definitions and compactness proper. Many of them are fully equivalent to the modern definition as long as we are operating within the usual topology of the real numbers. It is not surprising that early analysts, who did not venture too far from this setting, were content with their sequence-oriented formulation.

The main benefit of compactness is that it justifies reduction of the infinite to the finite in many important mathematical situations. Methods or arguments that are not immediately applicable in a given case may become appropriate after such a reduction step. The reader may gain an immediate appreciation of this idea by considering the following preview example. Suppose a property, like continuity or boundedness of a function, can be asserted to hold in a neighborhood of every point of a closed and bounded set of real numbers. We will discover that such sets are compact, and this allows us to conclude that the function is uniformly continuous or uniformly bounded on the set. Formulation of the Riemann integral, for example, depends on being able to do exactly this. Without compactness, conclusions of this type are generally unjustified.

Compactness is preserved by continuous functions, and we will exploit this fact repeatedly. Compact spaces have many other desirable properties, some not immediately apparent, which tend to make them particularly well characterized, and therefore valuable as landmarks or reference objects in more general settings.

Definition ***2.9.1*** *Let $(X, \mathcal{T})$ be a topological space. X will be called* ***compact*** *if every filter in X has an adherent point. A subset $K \subset X$ is compact if it is compact as a subspace. If the topology for which compactness holds requires emphasis, we can write* ***$\mathcal{T}$-compact****.*

Some, but by no means all, influential topology texts have incorporated the Hausdorff condition into the definition of compactness on the grounds that most applications of compactness involve Hausdorff spaces anyway. This has muddied the terminological waters somewhat, as they would refer to our compactness as **quasi-compactness**. As usual, awareness of the definitional convention in force will relieve confusion.

Example ***2.9.2*** *Subsets of $\mathbb{R}$, or more generally $\mathbb{R}^n$, are compact if and only if they are closed and bounded. This is the message of the Heine-Borel Theorem, and it depends on the* *special nature of $\mathbb{R}$, as the result is not necessarily true for more general spaces.*

Example *2.9.3 All products, without restriction, of compact factor spaces are compact by the famous Tychonoff Theorem.*

Example *2.9.4 All ordinal spaces, such as $[1,\omega]$ and $[1,\Omega]$, which are closed in the standard order topology are compact.*

Example *2.9.5 Any subsets of $C[a,b]$, the space of continuous real functions on the interval $[a,b]$ are compact for the topology of uniform convergence if and only if they are closed, bounded and (a little more than the Heine-Borel requirement) equicontinuous. This statement is the Arzela-Ascoli Theorem.*

Definition *2.9.6 Let $(X,\mathcal{T})$ be a topological space and $S \subset X$. S is* ***relatively compact*** *(with respect to X) if clS is compact.*

Example *2.9.7 The subsets $(0,1)$, $[0,1)$, and $(0,1]$ of $\mathbb{R}$ are relatively compact. We will see shortly that any subset of a compact space is relatively compact.*

We should emphasize that compactness, unlike closure, for example, is an absolute property of sets. This is as it must be for the definition of compact subsets to make sense. Consider the real interval $(0,1]$, which is closed in $(0,2]$ but not in $[0,1]$. By contrast, $[0,1]$ is compact as a subspace of any interval that happens to contain it. The relative compactness of a set involves a virtual closure, and therefore we would expect it to depend on the precise nature of the containing space.

Definition *2.9.8 Let X be a set and $\mathcal{F}$ a non-void family of sets in X such that any finite subfamily has nonvoid intersection. Then $\mathcal{F}$ is said to have the* ***finite intersection property (FIP)****.*

Any filterbase obviously has the FIP, but not every family of sets having the FIP is a filterbase. Consider, for example, the family of lines through the origin in $\mathbb{R}^2$. This family certainly has the FIP, but for it to constitute a filterbase, the origin would have to belong to the family.

Definition *2.9.9 Let $(X,\mathcal{T})$ be a topological space. Suppose $\{U_\alpha : \alpha \in A\}$ is a family of open sets for some index set A. If $\bigcup_{\alpha\in A} U_\alpha \supset X$, then $\{U_\alpha : \alpha \in A\}$ is an* ***open cover*** *of X. If $B \subset A$ and $\bigcup_{\alpha\in B} U_\alpha \supset X$, then $\{U_\alpha : \alpha \in B\}$ is called an* ***open subcover*** *of X, relative to the cover $\{U_\alpha : \alpha \in A\}$. In particular, if B is finite (resp. countable), then the corresponding family is called a* ***finite (resp. countable) subcover****. Substantially all of the covers we consider will be constructed from open sets, and we will routinely omit the word "open" when the context is clear. However, the notion of a* ***closed cover*** *is available, and is defined along with its variants as above, mutatis mutandi.*

Proposition *2.9.10 Let $(X,\mathcal{T})$ be a topological space. Then the following are equivalent:*

(i) X is compact
(ii) Every open cover of X has a finite subcover (Borel-Lebesgue property)
(iii) Every family of closed subsets of X with the FIP has nonvoid intersection
(iv) Every ultrafilter in X converges.

Proof $(i) \Rightarrow (ii)$ Suppose $\{U_\alpha : \alpha \in A\}$ is an arbitrary open cover of X for some index set

A, which we may assume to be infinite. For the sake of contradiction, assume no finite subcover exists. Consider the family of sets $\{X - \bigcup_{i \in H} U_i : A \supset H, \text{finite}\}$. By construction, the members of this family are nonvoid and the intersection of any two of them contains a third. This family is then a filterbase, which generates a filter having an adherent point x by hypothesis. Now $x \in cl(X - \bigcup_{i \in H} U_i)$ for every finite set H. But each set $X - \bigcup_{i \in H} U_i$ is already closed. It follows that $x \in X - \bigcup_{i \in H} U_i$ for every finite set H, and thus $x \in U_\alpha^c$ for every $\alpha \in A$. However this forces $x \in \bigcap_{\alpha \in A} U_\alpha^c = (\bigcup_{\alpha \in A} U_\alpha)^c = \varnothing$, which is absurd. It follows that compact sets display the Borel-Lebesgue property.

$(ii) \Rightarrow (iii)$ Suppose $\mathfrak{F}$ is an arbitrary family of closed sets in X with the FIP. For the sake of contradiction, say $\bigcap\{F \in \mathfrak{F}\} = \varnothing$. Then no point x of X is in every F, or equivalently, $x \in F^c$ for some $F \in \mathfrak{F}$. But then $\{F^c : F \in \mathfrak{F}\}$ is an open cover of X, which by hypothesis admits a finite subcover $\{F_i^c : F_i \in \mathfrak{F}, i \in H, \text{ finite}\}$. By complementation, $\bigcap_{i \in H} F_i^c = \varnothing$, which contradicts the FIP, and the result follows.

$(iii) \Rightarrow (iv)$ Suppose $\mathfrak{G}$ is an arbitrary ultrafilter in X. $\mathfrak{G}$ certainly has the FIP, since it is a filter, hence the family $\{clG : G \in \mathfrak{G}\}$ does as well. Then by hypothesis, $\bigcap_{G \in \mathfrak{G}} clG \neq \varnothing$, and there exists some $x \in adh\mathfrak{G}$. But then by §2.8.13, $\mathfrak{G} \to x$, as required.

$(iv) \Rightarrow (i)$ Suppose $\mathfrak{F}$ is an arbitrary filter in X. $\mathfrak{F}$ is refined by some ultrafilter $\mathfrak{G}$, which by assumption converges to a limit x. Then $x \in adh\mathfrak{G} \subset adh\mathfrak{F}$. By definition, X is compact. ■

Another less common characterization of compactness involves a special type of mapping, and this will be taken up after these mappings are introduced. In view of the Borel-Lebesgue condition, sets remain compact for coarser topologies.

Proposition ***2.9.11*** *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, $(X,\mathcal{T})$ compact and $f : X \to Y$ a continuous map. Then $f(X)$ is compact.*

Proof Suppose $\mathfrak{F}$ is a filter in $f(X)$. Then $\langle f^{-1}(\mathfrak{F})\rangle$ is a filter in X, which by compactness admits an adherence point x. But then $f(x) \in adh(\mathfrak{F})$, since any neighborhood of $f(x)$ must meet every $F \in \mathfrak{F}$ in order for x to adhere to $\langle f^{-1}(\mathfrak{F}\rangle$, and compactness of $f(X)$ is immediate. ■

Proposition ***2.9.12*** *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, $(Y,\mathcal{S})$ compact, and $f : X \to Y$ a closed injection. Then X is compact.*

Proof Suppose $\mathfrak{F}$ is an arbitrary filter in X., then $\mathfrak{F}' = \{clF : F \in \mathfrak{F}\}$ is as well. Consider $\langle f(\mathfrak{F}')\rangle$. Since Y is compact, there exists some $y \in adh\langle f(\mathfrak{F}')\rangle$. Since f is a closed map, $f(clF)$ is closed and hence $y \in f(clF)$, for each $F \in \mathfrak{F}$. By injectivity, $f^{-1}(y) \in f^{-1}(f(clF)) = clF$, and we see that $f^{-1}(y) \in adh\mathfrak{F}$. Since $\mathfrak{F}$ was arbitrary, X is compact. ■

Corollary ***2.9.13*** *Let $(X,\mathcal{T})$ be a compact space and $F \subset X$, with F closed. Then F is compact.*

Proof The inclusion map $inc : F \hookrightarrow X$ is closed, since F is closed, and §2.9.12 applies. ■

Example ***2.9.14*** *A subspace of a compact space need not be closed to be compact. Consider the Sierpinski topology $\{\varnothing, \{a\}, \{a,b\}\}$ on the space $\{a,b\}$. The set $\{a\}$ is not closed,* *but it is surely compact. If the Hausdorff condition is imposed on the space, then the implication of §2.9.13 can be reversed (see §2.9.20).*

Proposition ***2.9.15*** *Compactness is a topological property.*

Proof Let $(X, \mathcal{T})$ be a compact space, $(Y, \mathcal{S})$ a topological space, and $\Phi : X \to Y$ a homeomorphism. Since Φ is surjective, $Y = \Phi(X)$, and by §2.9.11, Y is compact. ■

Corollary ***2.9.16*** *Any continuous bijection on a compact space is a homeomorphism.*

Proof Let f be a continuous bijection. By injectivity, $(f^{-1})^{-1} = f$, hence by §2.9.12, $(f^{-1})^{-1}$ maps closed sets onto closed sets, and it follows that f is bicontinuous. ■

Proposition ***2.9.17*** *Let $(X, \mathcal{T})$ be a compact space and $\mathfrak{F}$ a filter in X. Then each neighborhood of $adh\mathfrak{F}$ contains a member of $\mathfrak{F}$.*

Proof Without restriction of generality, suppose that U is an open neighborhood of $adh\mathfrak{F}$. If $U = X$, there is nothing to prove, so assume $U^c \neq \emptyset$. If the assertion were false, each element of F must have nonvoid intersection with U^c, hence $\{F \cap U^c : F \in \mathfrak{F}\}$ is a filterbase in U^c. But U^c is closed, hence by §2.9.13 compact, and it follows that $\bigcap\{F \cap U^c : F \in \mathfrak{F}\} \neq \emptyset$. But then $adh(\mathfrak{F} \cap U^c) \subset adh(\mathfrak{F}) \subset U$, which is impossible, and the result follows. ■

Corollary ***2.9.18*** *Let $(X, \mathcal{T})$ be a compact space and $\mathfrak{F}$ a filter in X such that $adh\mathfrak{F}$ consists of the singleton x. Then $\mathfrak{F} \to x$.*

Proof By §2.9.17, $\mathfrak{F} \supset \mathcal{N}_x$, or $\mathfrak{F} \to x$. ■

Lemma ***2.9.19*** *Let $(X, \mathcal{T})$ be a Hausdorff topological space, $K \subset X$, K compact. If $y \in K^c$, then there are disjoint open sets U and V such that $K \subset U$ and $y \in V$.*

Proof If $K = X$ there is nothing to prove, so assume there exists $y \in X - K$. For each $x \in K$, by the Hausdorff condition, there exists a pair of open sets U_x and V_x, such that $x \in U_x$ and $y \in V_x$, with $U_x \cap V_x = \emptyset$. Now $\{U_x : x \in K\}$ is an open cover of K, hence by hypothesis, there exists a finite subcover $\{U_{x_i} : i \in H, \text{finite}\}$. Define $U = \bigcup_{i \in H} U_{x_i}$ and $V = \bigcap_{i \in H} V_{x_i}$. Clearly $K \subset U$, $y \in V$, and $U \cap V = \emptyset$. ■

Proposition ***2.9.20*** *Let $(X, \mathcal{T})$ be a Hausdorff topological space, $K \subset X$, K compact. Then K is closed.*

Proof If $K = X$ then K is closed. Otherwise, for each $y \in X - K$, the preceding lemma furnishes a set V_y such that $y \in V_y$ and $K \cap V_y = \emptyset$. Since $K^c = \bigcap_{y \in X-K} V_y$, and the latter is open, K must be closed. ■

Any finite indiscrete space with more than two points furnishes a counterexample to §2.9.20 in the absence of the Hausdorff condition. Every subset is compact, but the only nonvoid closed subset is the entire space.

Compact Hausdorff spaces are occasionally called **compacta**, especially in older texts. The combination of the Hausdorff condition and compactness places a surprisingly rigid constraint on admissible topologies for a compactum. In the lattice of possible topologies on such a space, any strictly coarser topology cannot be Hausdorff, and any strictly finer topology nullifies compactness.

Proposition ***2.9.21*** *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then:*

(i) If $\mathcal{S}$ is a Hausdorff topology for X, and $\mathcal{S} \subset \mathcal{T}$, then $\mathcal{S} = \mathcal{T}$
(ii) If X is $\mathcal{S}$-compact, and $\mathcal{S} \supset \mathcal{T}$, then $\mathcal{S} = \mathcal{T}$

Proof (i) Consider the map $id_X : (X, \mathcal{T}) \to (X, \mathcal{S})$, which, is continuous since $\mathcal{S} \subset \mathcal{T}$. By §2.9.16, $(X, \mathcal{T})$ and $(X, \mathcal{S})$ are homeomorphs., and the result is immediate.

(ii) Interchange $\mathcal{S}$ and $\mathcal{T}$ in (i). ■

Example ***2.9.22*** *In §4.4 we will define an operator $(\bullet)_S$ which assigns to each topological space $(X, \mathcal{T})$ a companion space $(X, \mathcal{T}_S)$, where $\mathcal{T}_S$ is the associated sequential topology. This topology is the finest one that can be imposed on X without destroying convergence of any sequence that converges relative to $\mathcal{T}$. In general, refinement only has the potential to reduce the family of convergent filters on a given space...it cannot create convergence where there is none. The idea here is to refine the given topology to the point where going further would begin to degrade the original family of convergent sequences. The closed sets of $\mathcal{T}_S$ are the sequentially closed sets of $\mathcal{T}$, and $\mathcal{T}_S$ refines $\mathcal{T}$. If $(X, \mathcal{T})$ is Hausdorff and a set $K \subset X$ is compact for $\mathcal{T}_S$, then the trace of $\mathcal{T}$ on K (i.e. $\{U \cap K : U \in \mathcal{T}\}$) is identical to the trace of $\mathcal{T}_S$ on K, since §2.9.21(ii) applies.*

Example ***2.9.23*** *Continuing with the leitmotif of the preceding example, let (x_n) be a sequence in X with $\mathcal{T}$-limit x. We will show in §4.4 that the $\mathcal{T}_S$-limit is also x. Now the set consisting of the range of any convergent sequence taken together with its limit is always compact, so by the prior example, the traces of $\mathcal{T}$ and $\mathcal{T}_S$ must agree on this set.*

We have now arrived at an appropriate moment to present an exceedingly important theorem on products of compact spaces. Its significance for functional analysis lies in the fact that a function space can be expressed as a product of suitably many copies of a range space, and if the range space happens to be compact, this theorem guarantees that the function space will be compact as well. The theorem was originally developed by Andrei Tychonoff in the mid-1930's with the restriction that only countably many factors could be taken in the product. An unrestricted version of the theorem was subsequently introduced, where the proof was based on using filters. Not only did filters allow generalization to arbitrary products, but they coincidentally reduced the complete proof to just a few lines. Filter fans often cite the following theorem as the premier example of the concept's utility.

Proposition ***2.9.24 (Tychonoff)*** *Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ be a family of topological spaces for some nonvoid index set A. Then the product $X = \prod_{\alpha \in A} X_\alpha$ is compact if and only if X_α is compact for all $\alpha \in A$.*

Proof (Necessity) Suppose each factor space is compact. Let $\mathfrak{F}$ be an arbitrary ultrafilter in X. Then $\langle \pi_\alpha(\mathfrak{F}) \rangle$ must be an ultrafilter in X_α for each $\alpha \in A$, otherwise the product filter $\prod_{\alpha \in A} \langle \pi_\alpha(\mathfrak{F}) \rangle$ in X would contain $\mathfrak{F}$ and admit a proper refinement, contrary to the specification of $\mathfrak{F}$. By §2.9.10(iv), there exists an $x_\alpha \in X_\alpha$ such that $\langle \pi_\alpha(\mathfrak{F}) \rangle \to x_\alpha$. It follows that $\mathfrak{F} \to (x_\alpha)_{\alpha \in A}$. Since $\mathfrak{F}$ was arbitrary, again by §2.9.10(iv), X is compact.

(Sufficiency) Suppose X is compact, then by §2.9.11, $\pi_\alpha(X) = X_\alpha$ is compact. ■

Corollary ***2.9.25*** *Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ be a family of topological spaces for some nonvoid index set A. Then the product $X = \prod_{\alpha \in A} X_\alpha$ is relatively compact if and only if X_α is* *relatively compact for all $\alpha \in A$.*

Proof (Necessity and Sufficiency) $clX = \prod_{\alpha\in A} clX_\alpha$ by §2.6.24. Hence by §2.9.24, clX_α is compact for each $\alpha \in A$ if and only if clX is compact, and the result is immediate. ■

Corollary ***2.9.26*** *Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ be a family of Hausdorff topological spaces for some nonvoid index set A. Then $S \subset X = \prod_{\alpha\in A} X_\alpha$ is relatively compact if and only if $\pi_\alpha(S)$ is relatively compact in X_α for each $\alpha \in A$.*

Proof (Necessity) If each $\pi_\alpha(S)$ is relatively compact in its factor space, then $cl\pi_\alpha(S)$ is compact, and by §2.9.24 $\prod_{\alpha\in A} cl\pi_\alpha(S)$ is compact in X. Note that X is Hausdorff by §2.7.15. By §2.6.24, we see that $\prod_{\alpha\in A} cl\pi_\alpha(S) = cl(\prod_{\alpha\in A} \pi_\alpha(S))$, which clearly contains clS. By §2.9.13, we conclude clS is compact, hence S is relatively compact.

(Sufficiency) $\pi_\alpha(clS)$ is compact in X_α by §2.9.11, hence closed by §2.9.20, so $\pi_\alpha(clS) \supset cl\pi_\alpha(S)$. By §2.5.9, $cl\pi_\alpha(S) \supset \pi_\alpha(clS)$, so it is apparent that $cl\pi_\alpha(S) = \pi_\alpha(clS)$, and hence $\pi_\alpha(S)$ is relatively compact. ■

One of the practical benefits of defining compactness without building in the Hausdorff condition is that the divisibility of compactness may be established without worrying about the parallel requirement that the Hausdorff condition be passed simultaneously to the quotient, which is generally not the case.

Proposition ***2.9.27*** *Let $(X, \mathcal{T})$ be a compact space, and R an equivalence relation on X. Then X/R is compact.*

Proof The natural map $q : X \to X/R$ is continuous, and the result follows from §2.9.11. ■

Corollary ***2.9.28*** *Let $(X, \mathcal{T})$ be a compact space, $f : X \to Y$ a continuous function, and R a relation on X such that $(x,y) \in R$ whenever $(f(x), f(y)) \in R$. Then $\hat{f} = f \circ q^{-1} : X/R \to Y$ is a homeomorphism.*

Proof R is easily seen to be an equivalence relation, hence by §2.9.27, X/R is compact. We claim $\hat{f}$ is continuous. To see this take U open in Y, then consider $f^{-1}(U)$, which is R-saturated (i.e. $f^{-1}(U)$ contains all $x \in X$ such that $(x,y) \in R$ for any $y \in U$). Hence $f^{-1}(U)$ is a q-saturated open set, which has an open image in X/R. Therefore $q \circ f^{-1}$ is an open map, and we conclude that its inverse $\hat{f}$ is continuous. Now $\hat{f}$ is also injective by construction and has range $f(Y)$. By §2.9.16, $\hat{f}$ is then a homeomorphism. ■

Corollary ***2.9.29*** *Let $(X, \mathcal{T})$ be a topological space, R a relation on X, and $S \subset X$, with S compact. If $q(X) = q(S)$, then X/R is compact.*

Proof The argument of §2.9.27 goes through for compact subsets of X, hence $q(S)$ is compact, and then by assumption, $q(X)$ is compact as well. ■

In §2.9.19 we saw that in a Hausdorff space, a compact set and a point not contained within it can be separated by a condition reminiscent of that for regularity. If the space itself is compact, more is true.

Proposition ***2.9.30*** *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(X, \mathcal{T})$ is regular.*

Proof Suppose F is closed in X and $x \notin F$. By §2.9.11, F is compact. By §2.9.12, x and F

can be separated by disjoint open sets. Since x and F were arbitrary, X is regular.

Alternate Proof: It is enough to exhibit a base of closed neighborhoods at an arbitrary point x. Since X is Hausdorff, by §2.7.10, $\bigcap_{N \in \mathcal{N}_x} clN = \{x\}$. But then the filter $\{clN : N \in \mathcal{N}_x\}$ has the single adherent point x. By §2.9.18, the filter converges to x, and hence must refine $\mathcal{N}_x$, giving the result. ■

Proposition ***2.9.31*** *Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(X, \mathcal{T})$ is normal.*

Proof Suppose F and G are disjoint closed subsets of X. By §2.9.11, F and G are compact. By §2.9.12, for each $y \in G$, there exists a pair of disjoint open sets U_y and V_y such that $F \subset U_y$ and $y \in V_y$. Now $\bigcup_{y \in G} U_y \supset G$, so $\{U_y : y \in G\}$ is an open cover of G, and by compactness, there exists a finite subcover $\{U_{y_i} : i \in H, \text{finite}\}$. Define $U = \bigcap_{i \in H} U_{y_i}$ and $V = \bigcup_{i \in H} V_{y_i}$. Clearly, U and V are open and disjoint, $F \subset U$, and $G \subset V$. Since F and G were arbitrary, it follows that X is normal. ■

The preceding result also holds if the space is not Hausdorff, but regular.

Proposition ***2.9.32*** *Let $(X, \mathcal{T})$ be a compact regular space. Then $(X, \mathcal{T})$ is normal.*

Proof Suppose F and G are disjoint closed subsets of X. For each $x \in F$, regularity gives disjoint open sets U_x and V_x such that $x \in U_x$ and $G \subset V_x$. Continuing as in §2.9.31, $\{U_x : x \in F\}$ is an open cover of F. Now F is compact by §2.9.13, hence there exists a finite subcover $\{U_{x_i} : i \in H, \text{finite}\}$. As before, define $U = \bigcap_{i \in H} U_{x_i}$ and $V = \bigcup_{i \in H} V_{x_i}$, and note that U and V are disjoint open sets with $F \subset U$ and $G \subset V$, which gives the result. ■

Compact sets enjoy their own version of functional normality in completely regular spaces, where Urysohn's Lemma, which makes the assumption of normality, does not necessarily apply.

Proposition ***2.9.33*** *Let $(X, \mathcal{T})$ be a completely regular space, $K \subset X$, K compact. If U is any open neighborhood of K, then there exists a continuous function $f : X \to [0, 1]$ with $f(K) = 1$ and $f(U^c) = 0$.*

Proof By assumption, for each $\alpha \in K$, there exists a continuous function $f_\alpha : X \to [0, 1]$ satisfying $f_\alpha(\alpha) = 1$ and $f_\alpha(U^c) = 0$. The set $N_\alpha = \{x : f_\alpha(x) > \frac{1}{2}\}$ is open in X. Note $\alpha \in N_\alpha$. Define a new function $g_\alpha(x) = 1 \vee 2f_\alpha(x)$ for each $\alpha \in K$. Each g_α is continuous, has range $[0, 1]$, and $g_\alpha(U^c) = 0$. Moreover, $g_\alpha(N) = 1$. Now the family $\{N_\alpha : \alpha \in K\}$ is an open cover of K, hence by compactness, there exists a finite subcover, say $\{N_{\alpha_i} : i \in H, \text{finite}\}$. Then the function $f(x) = \bigvee_{i \in H} g_{\alpha_i}(x)$ is continuous and satisfies $f(K) = 1$ and $f(U^c) = 0$. ■

The Borel-Lebesgue criterion for compactness need not be checked for an entire topology. Open covers using only basic sets or even subbasic sets are enough to determine compactness. The former result is straightforward, but the latter is much deeper, and is known as Alexander's Theorem.

Proposition ***2.9.34*** *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is compact if and only if for every base $\mathcal{B}$ and every open cover $\{U_\alpha \in \mathcal{B} : \alpha \in A, \text{arbitrary}\}$, there exists a finite open subcover.*

Proof (Necessity) Let $\{V_\beta \in \mathcal{T} : \beta \in B, \text{arbitrary}\}$ be an arbitrary open cover of X for some index set B. Here we are allowing the open sets to be drawn from the entire topology. Let $V_\beta = \bigcup_{\alpha \in A(\beta)} U_{\beta,\alpha}$, where $U_{\beta,\alpha} \in \mathcal{B}$ and $A(\beta)$ is a corresponding index set. Note $X = \bigcup_{\beta \in B} \bigcup_{\alpha \in A(\beta)} U_{\beta,\alpha}$, so $\{U_{\beta,\alpha} \in \mathcal{B} : \beta \in B, \alpha \in A(\beta)\}$ is an open cover of X using only basic sets. By hypothesis, there exists a finite subcover, say $\{U_i : i \in H,$ finite$\}$, where each U_i is a relabelled $U_{\beta,\alpha}$. Now for each U_i, we may select a corresponding V_β such that $U_i \subset V_\beta$. Relabel that V_β as V_i and note that $X = \bigcup_{i \in H} V_i$. In other words, $\{V_i : i \in H,$ finite$\}$ is a finite subcover, and it follows that X is compact.

(Sufficiency) §2.9.10(ii) obviously applies to covers with basic open sets. ■

Before we can prove the corresponding result for subbasic open sets, we need a few specialized notions. Our presentation of Alexander's Theorem follows [Kel55].

Definition *2.9.35 For any set X, call a family of subsets* $\{X_\alpha \subset X : \alpha \in A,$ *arbitrary*$\}$ ***inadequate*** *if* $\bigcup_{\alpha \in A} X_\alpha \neq X$. *Likewise, call the family* ***finitely inadequate*** *if* $\bigcup_{\alpha \in H} X_\alpha \neq X$ *if* $H \subset A$ *and H is finite.*

Definition *2.9.36 Let X be a set and* $\mathcal{F} \subset \wp(X)$. *Then* $\mathcal{F}$ *is said to have* ***finite character*** *if:*

(i) all finite subsets of any member of $\mathcal{F}$ are also contained in $\mathcal{F}$
(ii) any set, all of whose finite subsets belong to $\mathcal{F}$, belongs to $\mathcal{F}$

Expressed informally, a family of subsets has finite character if all finite subsets of a given set are present, and any other set whose finite subsets behave in this manner is also present. An infinite set can be in the family, as long as all its finite subsets are too.

§2.9.35 allows us to state the Borel-Lebesgue criterion in its contrapositive form, which seems a little odd at first glance: *X is compact if every finitely inadequate family of open sets in X is inadequate*. The collection of finitely inadequate families for a given set X has finite character, since every finite subfamily of any family is a fortiori inadequate, and any family, all of whose finite subfamilies are inadequate, is certainly finitely inadequate.

Remark *By a Zorn's Lemma argument, every family of finite character can be shown to be contained in a maximal family of finite character, so we immediately can say that every finitely inadequate family of sets in X is contained in a maximal such family.*

We need a technical lemma to pave the way:

Lemma *2.9.37 Let* $(X, \mathcal{T})$ *be a topological space, and* $\mathcal{F}$ *a maximal finitely inadequate family of open sets in X. Then:*

(i) If $U \in \mathcal{T}$ but $U \notin \mathcal{F}$, then no open set containing U belongs to $\mathcal{F}$.
(ii) If $\mathcal{G} \subset \mathcal{T}$, $\mathcal{G}$ is finite, and $\mathcal{G} \cap \mathcal{F} = \varnothing$, then no set containing $\bigcap\{G \in \mathcal{G}\}$ can belong to $\mathcal{F}$.

Proof (i) For the sake of contradiction, suppose $A \supset U$, where $A \in \mathcal{F}$. Any union from the arbitrary finite subfamily $\mathcal{F} \cup \{U\}$ would be contained in a corresponding union from the subfamily $\mathcal{F}$ by switching A for U in the union, if U appears. This latter subfamily does not cover X, since it is finitely inadequate, so neither does the former. But this means $\mathcal{F} \cup \{U\}$ is finitely inadequate, contrary to the hypothesis that $\mathcal{F}$ is maximal, therefore $A \notin \mathcal{F}$.

(ii) Some finite subfamily of $\mathcal{F}$, say $\{A_i \in \mathcal{F} : i \in H,$ finite$\}$ must exist, such that $U \cup \bigcup_{i \in H} A_i = X$, otherwise U could be adjoined to $\mathcal{F}$ without disturbing finite inadequacy, again contradicting the maximality of $\mathcal{F}$. If V is another open set not

contained in $\mathcal{F}$, then by a parallel argument there exists a subfamily $\{B_j \in \mathcal{F} : j \in J,$ finite$\}$ such that $V \cup \bigcup_{j\in J} B_j = X$. Then $(U \cup \bigcup_{i\in H} A_i) \cap \left(V \cup \bigcup_{j\in J} B_j\right) = X$ implies that
$(U \cap V) \cup \left(U \cap \bigcup_{j\in J} B_j\right) \cup (V \cap \bigcup_{i\in H} A_i) \cup \left((\bigcup_{i\in H} A_i) \cap \left(\bigcup_{j\in J} B_j\right)\right) \subset (U \cap V) \cup \bigcup_{i\in H}$
as well. Apparently $(U \cap V) \notin \mathcal{F}$, as the finite inadequacy of $\mathcal{F}$ would be contradicted yet again. This construction is clearly repeatable any finite number of times, and it follows that if a finite family of open sets $\mathcal{G}$ is given that does not overlap $\mathcal{F}$, then no set containing $\bigcap\{G \in \mathcal{G}\}$ can belong to $\mathcal{F}$. ■

Reinterpreted contrapositively, the lemma says that if $\bigcap\{G \in \mathcal{G}\} \subset W$ for some $W \in \mathcal{F}$, then $\mathcal{G} \cap \mathcal{F} \neq \emptyset$, or equivalently, a term of the finite intersection $\bigcap\{G \in \mathcal{G}\}$ belongs to $\mathcal{F}$. With these results in mind, let us consider Alexander's Theorem.

Proposition *2.9.38 (**Alexander**) Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is compact if and only if for every subbase $\mathcal{S}$ and every cover $\{U_\alpha \in \mathcal{S} : \alpha \in A$, arbitrary$\}$, there exists a finite subcover.*

Proof (Necessity) Assume $\mathcal{S}$ is a subbase for $\mathcal{T}$ such that each cover of X by members of $\mathcal{S}$ admits a finite subcover, or in now familiar terms, each finitely inadequate subfamily of $\mathcal{S}$ is also inadequate. Suppose that for some subfamily $\mathcal{R} \subset \mathcal{S}$ that $\mathcal{R}$ is finitely inadequate. We want to show that $\mathcal{R}$ is inadequate as well. By our preceding remark, $\mathcal{R}$ is contained in a maximal finitely inadequate subfamily of $\mathcal{S}$, call it $\mathcal{M}$. Now consider the family $\mathcal{S} \cap \mathcal{M}$, which must be finitely inadequate because $\mathcal{M}$ is, and also inadequate, because $\mathcal{S}$ is. By the base property, each point $x \in M \in \mathcal{M}$ is contained in some basic open set $O_x \subset M$, where $O_x = \bigcap\{S_i \in \mathcal{S} : i \in H$, finite$\}$. By the §2.9.37, it must be true that $S_i \in \mathcal{M}$ for some $i \in H$. But this amounts to saying that $\bigcup\{M \in \mathcal{M}\} = \bigcup\{M \in \mathcal{S} \cap \mathcal{M}\}$, and we now see that $\mathcal{M}$ is inadequate. Since $\mathcal{M}$ is inadequate, so is the arbitrary subfamily $\mathcal{R}$, and we conclude X is compact.

(Sufficiency) §2.9.10(ii) applies to covers by subbasic open sets. ■

The hard part of Tychonoff's Theorem, and the part which filters dispose of with almost indecent efficiency, is the necessity for the product to be compact if each factor space is compact. Alexander's Theorem gives us a different, but entirely general, proof of this.

Proposition *2.9.24′ (**Tychonoff / Alternate Proof of Necessity**)*

Proof With the notation of §2.9.24, let $\mathcal{S}$ be the canonical subbase for the product topology, i.e. all sets of the form $\pi_\alpha^{-1}(U_\alpha)$, where each U_α is open in X_α. Alexander's Theorem (§2.9.38) tells us that X is compact if every open cover of X by an arbitrary subfamily of $\mathcal{S}$ admits a finite subcover. Recall that contrapositively, X is compact if each subfamily $\mathcal{F}$ of $\mathcal{S}$, containing no finite subfamily that covers X, also does not cover X. Let $\mathcal{F} \subset \mathcal{S}$ be an arbitrary subfamily with this property. For $\alpha \in A$, let $\mathcal{G}_\alpha = \{U_\alpha \subset X_\alpha : \pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}\}$. Now if a finite subfamily of $\mathcal{G}_\alpha$ covered X_α, then the image of each member under π_α^{-1} would constitute a finite subcover of X, contrary to specification of $\mathcal{F}$. By the assumed compactness of X_α, $\mathcal{G}_\alpha$ must then not cover X_α, for each α. But then there exists some $x_\alpha \in X_\alpha$ such that $x_\alpha \notin U_\alpha$ for all $U_\alpha \in \mathcal{G}_\alpha$. This implies that there exists an $x = (x_\alpha)_{\alpha\in A}$ such that $x \notin G$, for all $G \in \mathcal{F}$. In other words, there must be room in each factor space for a point outside each cover, and then stringing these together in the product produces a point outside the cover of X. Hence the subfamily $\mathcal{F}$ does not cover X, and by the above formulation of the Borel-Lebesgue criterion, X is compact. ■

One prolific source of topological counterexamples is the realm of ordinal numbers. The following result establishes a condition for their compactness as a special case.

Proposition *2.9.39 Let X be a linearly ordered set which is order complete. Then, for the standard order topology on X, each closed interval in X is compact.*

Proof Let $[x,y]$ be a closed order interval in X, and suppose $\mathfrak{F}$ is any filter in $[x,y]$. Define $\alpha(F) = \sup\{x : x \in clF, F \in \mathfrak{F}\}$ and $\alpha = \inf\{\alpha(F) : F \in \mathfrak{F}\}$. Claim: $\alpha \in clF$ for all $F \in \mathfrak{F}$. Suppose not. Then $\alpha \notin clG$ for some $G \in \mathfrak{F}$. It follows that there exists a basic order neighborhood (ζ,ξ) containing α and disjoint from clG. Also, by definition of α, there exists an $H \in \mathcal{F}$ with $\alpha(H) \in (\zeta,\xi)$. $G \cap H \neq \varnothing$ by the filter condition, but then $\alpha(G \cap H) \leq \zeta < \alpha$, contrary to the specification of α, and the claim is seen to be valid. Thus $\alpha \in \bigcap\{clF : F \in \mathfrak{F}\} = adh\mathfrak{F}$, and since $\mathfrak{F}$ was arbitrary, $[x,y]$ is compact. ■

Closed intervals in ordinal spaces are therefore compact, and in particular, the spaces $[1,\omega]$ and $[1,\Omega]$ are compact. We will be able to easily prove the Heine-Borel Theorem with the aid of §2.9.39.

An interesting consequence of compactness is that perfect sets in compact Hausdorff spaces are intrinsically uncountable. Recall that a perfect set is one which has no isolated points. To show this we need a lemma that is a generalization of a result originally proved by Cantor for nested intervals of real numbers.

Proposition *2.9.40 (**Cantor**) Let $(X,\mathcal{T})$ be a topological space. If $(C_n)_{n\in\mathbb{N}}$ is a sequence of closed sets such that $C_n \supset C_{n+1}$ and C_N is compact for some $N \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$.*

Proof The sets C_m for $m \geq N$ constitute a filter $\mathfrak{F}$ in C_N, which by compactness has nonvoid adherence. But then there exists some $x \in clC_m$ for $m \geq N$. However, the sets C_m are closed, hence $x \in C_m$ for $m \geq N$, and then by assumption, for $m \geq 1$. It follows that $x \in \bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$. ■

Proposition *2.9.41 Let $(X,\mathcal{T})$ be a nonvoid compact Hausdorff space, with $A \subset X$. If A is perfect, then $card(A) > \aleph_0$.*

Proof Since A is perfect, A is closed, hence compact by §2.9.13. Claim: given any $x \in A$ and nonvoid open set $U \subset A$, there exists an open set $V \subset U$ such that $x \notin clV$. If $x \in U$, then since x is a limit point of A, there exists some $y \in U$ distinct from x. Similarly, if $x \notin U$, then since U is nonvoid, there is a $y \in U$. By the Hausdorff property, there are disjoint open neighborhoods W_1 and W_2 containing x and y, respectively. The set $V = W_2 \cap U$ satisfies the claim, since $clW_2 \cap W_1 = \varnothing$. Now relabel A as V_0, x as x_1, and call the set determined in the claim V_1. Clearly, clV_1 is compact, again by §2.9.13. We may pick an $x_2 \in clV_1$, which must obviously be distinct from x_1, and repeat the argument for V_1 to obtain an open set $V_2 \subset V_1$ such that $x_2 \notin V_2$. It follows that this construction can be repeated any finite number of times, and results in a sequence of distinct points $(x_n)_{n\in\mathbb{N}}$ and open sets $(V_n)_{n\in\mathbb{N}}$ satisfying the conditions that $x_n \notin clV_n$ and $clV_n \supset clV_{n+1}$. Since $(clV_n)_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets, by §2.9.40 there exists some $z \in \bigcap_{n\in\mathbb{N}} clV_n$. By construction, $z \neq x_n$ for any $n \in \mathbb{N}$. Reviewing our construction, we see that we are free to pick the points x_n as we please. If A were countable, let $(x_n)_{n\in\mathbb{N}}$ be an enumeration of A. Then $z \in A$ but $z \notin \{x_n\}_{n\in\mathbb{N}}$. The contradiction establishes the uncountability of A. ■

Compactness may also be characterized by a class of maps that are distinguished by their behavior as factors of product maps.

Definition ***2.9.42*** *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, and $f : X \to Y$ a continuous map. If $f \times id_Z : X \times Z \to Y \times Z$ is a closed map for every topological space $(Z,\mathcal{R})$, then f is called a* ***proper map****.*

In this definition, id_Z is, of course, a closed map. It may seem superficially plausible that any continuous *f*, which is likewise closed, would qualify as a proper map, since then the product map would be closed, as required. Proper maps turn out to be necessarily closed, but unfortunately the product of closed maps need not be closed.

Proposition ***2.9.43*** *Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, and $f : X \to Y$ a proper map. Then f is a closed map.*

Proof Let $Z = \{z\}$, a singleton space. Suppose $F \subset X$ is closed, then $F \times \{z\}$ is closed, and by assumption, $f \times id_Z : F \times \{z\}$ is closed, too. But this must be the same as $cl(F \times \{z\}) = clF \times \{z\}$. Evidently $f(F) = clf(F)$, and we see that *f* is a closed map. ■

There are closed continuous maps which are not proper. The following example demonstrates this and illustrates, coincidentally, the preceding assertion that the product of closed maps need not be closed.

Example ***2.9.44*** *Let $f : \mathbb{R} \to \mathbb{R}$ be the constant map $x \mapsto 0$. f is surely closed and continuous. The map $f \times id_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ is not proper, since $f \times id_{\mathbb{R}^2} = \pi_y$, the projection onto the y-axis, which is not a closed map. To see this, consider $\pi_y(\{(x,y) : xy = 1\}) = (0,\infty)$.*

We will show in a moment that compact spaces can be characterized by the fact that constant maps on them are proper. Some texts actually use this as the primitive definition of compactness, and then immediately derive the equivalent conditions in terms of filters and the Borel-Lebesgue criterion. To show this we need a few specialized terms.

Definition ***2.9.45*** *Let X be an arbitrary set, and $\mathcal{F} = \{\mathfrak{F}_x : x \in X\}$ a family of filters in X such that $\langle x \rangle$ refines $\mathfrak{F}_x$ for each $x \in X$. $\mathcal{F}$ is called a* ***coherent family of filters*** *if for every $x \in X$ and $F \in \mathfrak{F}_x$ there exists some $G \in \mathfrak{F}_x$ such that $F \in \mathfrak{F}_y$ for every $y \in G$.*

The coherence condition may seem vaguely familiar. It is the precursor of Hausdorff's fourth axiom (see §2.1.15), which we might properly call the coherence axiom. The first three Hausdorff axioms postulate the existence of neighborhood filters at each point of a space. The coherence axiom weaves those filters together consistently. Coherence accounts for the reality that neighborhood systems of points in a topological space cannot be completely arbitrary, but must be coördinated in some manner. After all, a neighborhood of a point stands in *some* relation to the neighborhood filters of all the points it contains. The coherence axiom stipulates the existence of a set which is more than just a neighborhood of a given point, but a neighborhood of *each of its own points* as well. Clearly, the collection of sets with this property turns out to be the topology induced by the four Hausdorff axioms. We have something less than that in §2.9.45, but we can recover the coherence axiom from it. The requirement that the principal filter at each point refine each filter in the coherent family ensures that the point *x*, which tags each filter, also adheres to the tagged filter.

Proposition ***2.9.46*** *Let X be an arbitrary set, and $\mathcal{F} = \{\mathfrak{F}_x : x \in X\}$ a coherent family of filters in X. Then each member of the coherent family satisfies Hausdorff's fourth axiom (H4 in*

§2.1.15).

Proof Given an arbitrary $\mathfrak{F}_x \in \mathcal{F}$ we must show that each $F \in \mathfrak{F}_x$ contains some $U \in \mathfrak{F}_x$, depending on F, of course, with the property that U is a member of $\mathfrak{F}_y$ for every $y \in U$. So given F, we set $U = \{y \in X : F \in \mathfrak{F}_y\}$. Clearly $x \in U$ because $x \in \mathfrak{F}_x$. Also, $F \in \mathfrak{F}_y$ implies $y \in F$, so $U \subset F$. Now if $y \in U$, then $F \in \mathfrak{F}_y$, and by the hypothesis of coherence, there exists some $G \in \mathfrak{F}_y$ such that $F \in \mathfrak{F}_z$ for each $z \in G$. But $F \in \mathfrak{F}_z$ forces $z \in U$. It follows that $G \subset U$, and by the filter property, evidently $U \in \mathfrak{F}_y$. Note that $x \in U$ implies $U \in \mathfrak{F}_x$. Since F was arbitrary, the result follows. ■

Corollary ***2.9.47*** *Let X be an arbitrary set, and $\mathcal{F} = \{\mathfrak{F}_x : x \in X\}$ a coherent family of filters in X. Then $\mathcal{F}$ determines a topology on X for which the neighborhood system at each x corresponds to the filter $\mathfrak{F}_x$.*

Proof Since each filter $\mathfrak{F}_x$ in the coherent family is refined by the principal filter $\langle x \rangle$, axiom H1 in §2.1.15 is satisfied. Axioms H2 and H3 are automatically satisfied by the filter property of each $\mathfrak{F}_x$, and the preceding proposition establishes conformance with H4. The result is then immediate from §2.1.17. ■

A coherent family of filters on a set X constitutes a **fundamental system of neighborhoods** for the topology induced by the family.

Definition ***2.9.48*** *Let $(X, \mathcal{T})$ be a topological space, and $\mathfrak{F}$ an arbitrary filter in X. Adjoin any point $\alpha \notin X$ to X to form $X' = X \cup \{\alpha\}$. Define $\mathfrak{F}' = \{F \cup \{\alpha\} : F \in \mathfrak{F}\}$. The family of filters $\mathcal{F} = \{\mathfrak{F}'\} \cup \{\langle x \rangle : x \in X\}$ is a coherent family, and hence constitutes a fundamental system of neighborhoods for a topology $\mathcal{T}'$ on X'. The space $(X', \mathcal{T}')$ is called the **topological space associated with the filter** $\mathfrak{F}$.*

Proposition ***2.9.49*** *Let $(X, \mathcal{T})$ and $(\{p\}, \mathcal{I})$ be topological spaces, where p is a single point. Then X is compact if and only if every mapping $f : X \to \{p\}$ is proper.*

Proof (Necessity) Suppose f is proper. Let $\mathfrak{F}$ be an arbitrary filter in X, and define $X' = X \cup \{\alpha\}$, and let $(X', \mathcal{T}')$ be the topological space associated with $\mathfrak{F}$. Now consider the "diagonal" $\Delta = \{(x,x) : x \in X\}$. By the properness of f, $f \times id_{X'} : X \times X' \to \{p\} \times X'$ is a closed map, hence $f \times id_{X'}(cl\Delta)$ is closed in $\{p\} \times X'$. Since $\alpha \in cl_{\mathcal{T}'}X$, it must be that $(p, \alpha) \in f \times id_{X'}(cl\Delta)$. Evidently there exists a point $(x, \alpha) \in cl\Delta$ with $(f \times id_{X'})^{-1}(p, \alpha) = (x, \alpha)$. It follows that for every $N \in \mathcal{N}_x$ and $F \in \mathfrak{F}$, $(N \times F) \cap \Delta \neq \emptyset$. But this implies that $N \cap F \neq \emptyset$. We conclude that $x \in clF$ for all $F \in \mathfrak{F}$, and this means $adhF \neq \emptyset$. Since $\mathfrak{F}$ was arbitrary, X must be compact.

(Sufficiency) Suppose X is compact. Constant maps are continuous, so for f to be proper, it is enough to show that that for every topological space Z, $f \times id_Z : X \times Z \to \{p\} \times Z$ is closed. Suppose $F \subset X \times Z$ is closed, and an arbitrary point $w \in (p,z)$ adheres to $f \times id_Z(F) \neq \emptyset$. This requires that every element of the preimage filter $\mathfrak{N} = (f \times id_{X'})^{-1}(\mathcal{N}_w)$ have nonvoid intersection with F, and in turn, this is sufficient to conclude that the trace filter $\mathfrak{N}|_F = \{N \cap F : N \in \mathfrak{N}\}$ exists. Also, $\langle \pi_X(\mathfrak{N}) \rangle$, the direct image filter induced by the first projection, has nonvoid adherence by the compactness of X, say $x \in adh\langle \pi_X(\mathfrak{N}) \rangle$. Claim: $(x,z) \in F$. If not, since F is closed, there exists a product open set $V = V_X \times V_Z \subset X \times Y$ such that $(x,z) \in V$, where $V \cap F = \emptyset$. Clearly, both $X \times V_Z$ and $V_X \times Z$ belong to $\mathfrak{N}|_F$. But as elements of the trace filter, $((X \times V_Z) \cap F) \cap ((V_X \times Z) \cap F) = (V_X \times V_Z) \cap F = V \cap F \neq \emptyset$, which is absurd.

The contradiction establishes the claim, and it follows that $w = f \times id_Z(x,z)$ is contained in $f \times id_Z(F)$, hence $f \times id_Z$ is closed, and then f is proper. ■

With the notation of the preceding proposition, we see that $\{p\} \times Z \cong Z$, and compactness could be rephrased in terms of projections. A space X is compact if for every topological space Z, the projection "parallel" to the space in question $\pi_Z : X \times Z \to Z$ is closed. This seems mildly paradoxical at first glance...the compactness of X seems to depend on the arbitrary space Z. The properties of X, however, assert themselves through the product to the extent necessary to characterize compactness.

Definition *2.9.50 A proper map which is also surjective is called a* ***perfect map****.*

Definition *2.9.51 Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, and $f : X \to Y$ a map.*

(i) If for all $K \subset Y$, $f^{-1}(K)$ is compact, then f is called a compact map.
(ii) If for all $K \subset X$, $f(K)$ is compact, then f is called a compact-preserving map.
(iii) If for all $y \in Y$, $f^{-1}(y)$ is compact, then f has the compact point inverse (CPI) property.

The terminology here is unsettled. Some texts require proper maps to be surjections, and hence merge with what we are calling perfect maps. §2.9.53 presents a characterization of proper maps that is sometimes used to define them. Perfect maps acquired their name on the basis of being able to export many of the topological properties from domain to range and vice versa without being homeomorphisms. Of course, the cynic might protest that *really* perfect maps would be homeomorphisms.

Lemma *2.9.52 Suppose $f : X \to Y$ is closed, continuous, and has the CPI property. Let $\mathfrak{F}$ be a filter in X such that $y \in adh\langle f(\mathfrak{F})\rangle$. Then there exists an $x \in X$ such that $y = f(x)$.*

Proof Consider the filter $\mathfrak{H} = \{clF : F \in \mathfrak{F}\}$. By §2.5.9, $f(clF) \subset clf(F)$, so by the closedness of f we conclude that $f(clF) = clf(F)$. Since $y \in clf(F)$ for each $F \in \mathfrak{F}$, we have $f^{-1}(y) \subset clF$, or that $f^{-1}(y) \cap clF \neq \emptyset$ for all $F \in \mathfrak{F}$. Then $\{f^{-1}(y) \cap clF : F \in \mathfrak{F}\}$ is a filterbase in $f^{-1}(y)$ consisting of closed sets. Since $f^{-1}(y)$ is compact, by §2.8.11 there exists some $x \in adh\{f^{-1}(y) \cap clF : F \in \mathfrak{F}\}$, from which it follows that $x \in f^{-1}(y)$, establishing the result. ■

Proposition *2.9.53 Let $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ be topological spaces, and $f : X \to Y$ a continuous map. Then f is proper if and only if f is closed and has the CPI property.*

Proof (Necessity) We need only show that if Z is any topological space, then $f \times id_Z : X \times Z \to Y \times Z$ is a closed map. We will use the continuity of the product map and apply §2.8.22. So consider an ultrafilter $\mathfrak{G}$ in $X \times Z$ such that the ultrafilter generated by $f \times id_Z(\mathfrak{G})$ converges to some $(y,z) \in Y \times Z$. Then the projections $\langle f(\mathfrak{G})\rangle$ and $\langle id_Z(\mathfrak{G})\rangle$ generate ultrafilters in Y and Z that converge to y and z, respectively. By assumption, f is closed, continuous, and has the CPI property. So by §2.9.52, there must exist an $x \in X$ such that $y = f(x)$. Likewise, trivially, for id_Z, and it follows that $\mathfrak{G} \to (x,z)$. Evidently $f \times id_Z$ is closed by §2.8.22, hence f is proper, as required.

(Sufficiency) Suppose f is proper. The result is vacuously true for all $y \notin f(X)$, so assume that $y \in f(X)$. Let $A = f^{-1}(y)$ and consider the restriction of f to A. Claim: $f|_A$ is a proper map. Suppose Z is any topological space, then by hypothesis, $f \times id_Z$ is a closed map. Now $f|_A \times id_Z = (f \times id_Z)|_{A\times Z}$, and $A \times Z$ is closed, since A is a point inverse under f, which is continuous by definition. We remark that the restriction of a closed map to a closed set is closed. For if we are given $F \subset A$ closed in the relative topology, then

$F = C \cap A$, where C is closed in X. Then F is closed in X and $f(F)$ is closed in Y, by assumption. Since $f|_A(F) = f(F)$, the remark follows and we see that $(f \times id_Z)|_{A \times Z}$ is a closed map, establishing the claim. Since the map $f|_A : A \to \{y\}$ is proper, by §2.9.49, A is compact. This establishes the point inverse condition for f, since y was arbitrary. ■

We need the following lemma to establish that all perfect maps are compact maps.

Lemma *2.9.54 Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and $f : X \to Y$ a continuous surjection. Then f is a closed map if and only if for each point inverse $f^{-1}(y)$, $y \in Y$ and each $U \in \mathcal{T}$, $f^{-1}(y) \in U$ implies that there exists some $V \in S$ such that $f^{-1}(V) \subset U$.*

Proof (Necessity) Suppose $F \subsetneq X$ is an arbitrary closed set. It is enough to show that $f(F)$ is closed in Y. For the sake of contradiction, suppose not. Then there exist some $y \notin f(F)$ such that $y \in clf(F)$. By surjectivity, $f^{-1}(y) \neq \emptyset$. If $f^{-1}(y) \cap F \neq \emptyset$, then there exists an $x \in F$ such that $f(x) = y \in F$, which is absurd. So assume $f^{-1}(y) \cap F = \emptyset$, and take F^c as an open neighborhood of $f^{-1}(y)$. By hypothesis, there exists an open set $V \in \mathcal{N}_y$ such that $f^{-1}(y) \in F^c$. But y is alimit point of $f(F)$, hence $f^{-1}(y) \cap F \neq \emptyset$, which is impossible. The contradiction establishes that f is a closed map.

(Sufficiency) Suppose f is closed, and for an arbitrary $y \in Y$, $f^{-1}(y) \subset U \in \mathcal{T}$. Then U^c is closed, and by assumption, $f(U^c)$ is closed in Y. Let $V = (f(U^c))^c$, and note that V is open and $y \in V$. It remains to be shown that $f^{-1}(V) \subset U$. But since $f^{-1}(f(U^c)) \supset U^c$, it follows at once that $(f^{-1}(f(U^c)))^c$, or $f^{-1}(V) \subset U$, as required. ■

Proposition *2.9.55 Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces, and $f : X \to Y$ a perfect map. Then f is a compact map.*

Proof Suppose $K \subset Y$ is compact, and consider an arbitrary open cover $\{U_\alpha : U_\alpha \in \mathcal{S}, \alpha \in A\}$ of $f^{-1}(K)$. We will show that a finite subcover exists. Since f is perfect, hence proper, by §2.9.53 f has the CPI property. Now for each $y \in K$, $\{U_\alpha : U_\alpha \in \mathcal{S}, \alpha \in A\}$ covers $f^{-1}(y)$. Accordingly, there exists a finite subcover $\{U_{y,i} : i \in H_y\}$ where H_y is a finite index set depending on y. Define $V_y = \bigcup_{i \in H_y} U_{y,i}$. We know that f is a closed, continuous surjection, hence by §2.9.54, there exists an open set W_y such that $y \in W_y$ and $f^{-1}(W_y) \subset V_y$, for each $y \in K$. Now $\{W_y : y \in K\}$ is an open cover of K, so by compactness, there exists a finite subcover $\{W_j : j \in J, \text{finite}\}$. From the condition $f^{-1}(W_y) \subset V_y$ we can enumerate a corresponding finite cover of the form $\{V_j : j \in J\}$. And since each V_j is itself a union of finitely many of the original sets U_α, we have identified a finite subcover of $\{U_\alpha : U_\alpha \in \mathcal{S}, \alpha \in A\}$, establishing that f is a compact map. ■

We now turn to a circle of ideas that are allied with compactness, and in some cases within restricted settings equivalent to it.

Definition *2.9.56 Let X be any set and $f : X \to \mathbb{R}$ a real-valued function. We say that f is* ***bounded*** *on a subset $T \subset X$ if there exists some number M such that $t \in T$ implies $|f(t)| < M$. We denote the collection of bounded real-valued functions on the set X by $B(X)$.*

We will have occasion to generalize this somewhat, but in its present form, our definition of boundedness can be used to define pseudocompactness.

Definition *2.9.57 Let $(X, \mathcal{T})$ be a topological space. We call X* ***pseudocompact*** *if every* *continuous function $f : X \to \mathbb{R}$ is bounded on X, or in other words $C(X) \subset B(X)$.*

Pseudocompactness is another property preserved by continuous functions, as the following analog of §2.9.11 shows.

Proposition ***2.9.58*** *Let $(X, \mathcal{T})$ be a pseudocompact space, $(Y, \mathcal{S})$ a topological space, and $f : X \to Y$ a continuous function. Then $f(X)$ is pseudocompact.*

Proof Given an arbitrary $g \in C(f(X))$, $g \circ f$ is continuous on X and, by pseudocompactness, bounded. But then $g : f(X) \to \mathbb{R}$ is bounded, and since g was arbitrary, $f(X)$ must be pseudocompact as well. ■

In the introduction to this section, we alluded to compactness-like properties that were not fully equivalent to compactness, but behaved like it in certain restricted settings. Two of these are countable compactness, which mimics the Borel-Lebesgue criterion except for a countability condition on the open cover, and limit point compactness, which requires all infinite subsets of a given space to be suitably organized so that limit points are forced to appear.

Definition ***2.9.59*** *Let $(X, \mathcal{T})$ be a topological space. If every countable open cover of X admits a finite subcover, then we call X* ***countably compact****. A subset $Y \subset X$ is countably compact if it is countably compact in its own right as a subspace.*

Definition ***2.9.60*** *Let $(X, \mathcal{T})$ be a topological space. If every infinite subset $A \subset X$ admits a limit point, then we call X* ***limit point compact****. Note that the limit point required to exist need not be in A. Synonyms for limit point compact are* ***Bolzano-Weierstrass (BW) compact*** *and* ***Fréchet compact****.*

Example ***2.9.61*** *The space of all countable ordinal numbers is not compact. This does not conflict with §2.9.39, since the space is not closed. However this space is both countably compact and limit point compact.*

Example ***2.9.62*** *Consider the* ***transfinite line*** *(“****long line****”) $L = [1, \Omega] \times [0, 1)$. L is endowed with the lexicographic order, and the corresponding standard order topology. It turns out that L is not compact, but surprisingly enough, it is both countably compact and limit point compact.*

Countable compactness can be characterized by filters, and this is occasionally used as a definition.

Proposition ***2.9.63*** *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is countably compact if and only if every filter in X with a countable filterbase has nonvoid adherence.*

Proof (Necessity) Let $\{U_i : i \in \mathbb{N}\}$ be a countable open covering of X. For the sake of contradiction, suppose no finite subcover exists. Consider the family of sets $\{X - \bigcup_{i \in H} U_i : \mathbb{N} \supset H, \text{finite}\}$. The members of this family are nonvoid and the intersection of any two is also a set of this form, so they constitute a filterbase for a filter in X. There are countably many finite subsets of $\mathbb{N}$, hence this family is countable, and then by assumption, it must have an adherent point, say x. Now $x \in cl\{X - \bigcup_{i \in H} U_i\}$ and since the set $X - \bigcup_{i \in H} U_i$ is already closed, it follows that $x \in X - \bigcup_{i \in H} U_i$ for every finite index set H. We then conclude that $x \in U_i^c$ for every $i \in \mathbb{N}$. But then

$x \in \bigcap_{i\in\mathbb{N}} U_i^c = (\bigcup_{i\in\mathbb{N}} U_i)^c = \varnothing$, which is absurd. The contradiction implies the existence of a finite subcover, hence X is countably compact.

(Sufficiency) Suppose $(X, \mathcal{T})$ is countably compact and $\mathfrak{F}$ is a filter with a countable filterbase $\mathfrak{B} = \{B_i : i \in \mathbb{N}\}$. We must show that $adh\mathfrak{F} \neq \varnothing$, which will follow if $adh\mathfrak{B} \neq \varnothing$. For the sake of contradiction, suppose $adh\mathfrak{B} = \varnothing$. Then $\bigcap_{i\in\mathbb{N}} clB_i = \varnothing$, or dually, $\bigcup_{i\in\mathbb{N}}(clB_i)^c = X$. By hypothesis, $\bigcup_{i\in H}(clB_i)^c = X$ for some finite index set $H \subset \mathbb{N}$. It follows that $\bigcap_{i\in H} clB_i = \varnothing$, and in turn we have $\bigcap_{i\in H} B_i = \varnothing$, which contravenes the finite intersection stability of $\mathfrak{B}$. From the contradiction we conclude that $adh\mathfrak{F} \neq \varnothing$, as required. ■

Countably compact spaces are always limit point compact, but not vice versa, in general.

Example ***2.9.64*** *Consider $\mathbb{R}$ with the topology generated by the base $\{B_r = (-r, r) : r \in \mathbb{R}_+\}$. The family of sets $\{(-n, n) : n \in \mathbb{N}\}$ is a countable open cover of $\mathbb{R}$, but no finite subcover exists. On the other hand, if A is an infinite subset of $\mathbb{R}$, choose some $t \in A$. Then $-t$ is a limit point of A, since for all B_r such that $-t \in B_r$, we have $t \in (B_r - \{-t\}) \cap A$. Evidently $\mathbb{R}$ is limit point compact for this topology.*

The T_1 property is enough to collapse the difference between countable compactness and limit point compactness.

Proposition ***2.9.65*** *Let $(X, \mathcal{T})$ be a topological space. If $(X, \mathcal{T})$ is countably compact, then it is limit point compact. Moreover, if $(X, \mathcal{T})$ is a T_1 space, then the converse also holds.*

Proof (Necessity) Suppose X is T_1 and limit point compact. If, to the contrary, X is not countably compact, there exists an open cover $\{U_i : i \in \mathbb{N}\}$ of X with no finite subcover. This allows us to select for every n an $x_n \in X - \bigcup_{i=1}^n U_i$. Suppose the set $\{x_n\}_{n\in\mathbb{N}}$ has the limit point x, given by assumption. Now $x \in U_{i_x}$ for some $i_x \in \mathbb{N}$ by the covering property. By construction, $x_n \in U_{i_x}$ for only finitely many indices, say $n \leq N$. By the T_1 property, there exist open sets V_n such that $x \in V_n$ and $x_n \in V_n^c$ for all $n \leq N$. Then $U_{i_x} \cap \left(\bigcap_{n\leq N} V_n\right)$ is open, contains x, and has void intersection with $\{x_n\}_{n\in\mathbb{N}}$. This contradicts limit point compactness, and we conclude that X must be countably compact.

(Sufficiency) Suppose X is countably compact, and assume that $A \subset X$ is infinite. We clearly may form a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct elements of A. The elementary filter $\mathfrak{F}$ associated with this sequence has a countable base and therefore nonvoid adherence by §2.9.63, say $x \in adh\mathfrak{F}$. Now x is a limit point of the set $\{a_n\}_{n\in\mathbb{N}}$, hence also a limit point of A, establishing the result. ■

Proposition ***2.9.66*** *Let $(X, \mathcal{T})$ be a second countable, countably compact space. Then $(X, \mathcal{T})$ is compact.*

Proof Suppose $\{U_\alpha : \alpha \in A\}$ is an arbitrary open cover of X. By assumption, there exists a countable base $\mathcal{B}$ for $\mathcal{T}$. For each $x \in X$, choose some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_\alpha$ for some α. Now the family $\{B_x \in \mathcal{B} : x \in X\}$ is obviously a countable open cover of X, and by assumption, it contains a finite subcover, say $\{B_i : i \in H, \text{finite}\}$. If $B_i \subset U_\alpha$, relabel U_α as U_i. Then $\{U_i : i \in H\}$ is a finite subcover of X derived from the original arbitrary open cover, and it follows that X is compact. ■

We will postpone stating the obvious corollary, which weaves together the various compactness notions, until we have discussed the Lindelöf condition and sequential compactness. The Lindelöf condition is a weaker cousin of the Borel-Lebesgue criterion. It reduces arbitrary open covers to countable, but not finite, subcovers.

Definition *2.9.67 Let $(X, \mathcal{T})$ be a topological space. If every open cover of X admits a countable subcover, then we call X a a* ***Lindelöf*** *space. A subset $Y \subset X$ is Lindelöf if it is Lindelöf in its own right as a subspace.*

Example *2.9.68 The positive integers with the discrete topology are a trivially noncompact Lindelöf space.*

Example *2.9.69 $\mathbb{R}$ with its usual topology is noncompact but Lindelöf.*

Proposition *2.9.70 Let $(X, \mathcal{T})$ be a countably compact Lindelöf space. Then $(X, \mathcal{T})$ is compact.*

Proof By the Lindelöf property, any open cover is reducible to a countable subcover, which in turn, by countable compactness, is reducible to a finite subcover. But this is the Borel-Lebesgue criterion, and the result is clear. ■

Because of their parallel definitions, many arguments for countably compact and Lindelöf spaces are of the mutatis mutandi variety.

Proposition *2.9.71 Let $(X, \mathcal{T})$ be a countably compact (resp. Lindelöf) space, $(Y, \mathcal{S})$ a topological space, and $f : X \to Y$ a continuous function. Then $f(X)$ is countably compact (resp. Lindelöf).*

Proof (Countably compact case) Let $\{U_i : i \in \mathbb{N}\}$ be a countable open cover of $f(X)$. Then the preimages $\{f^{-1}(U_i) : i \in \mathbb{N}\}$ constitute a countable open cover of X. By assumption, there exists a finite subcover, say $\{f^{-1}(U_i) : i \in H, \text{finite}\}$, but then $\{U_i : i \in H\}$ covers $f(X)$, which is seen to be countably compact. The Lindelöf case goes through mutatis mutandi. ■

Limit point compactness is not preserved by continuous functions, although in view of §2.9.65 and the preceding, any counterexample would have to involve a non-T_1 space.

Proposition *2.9.72 Countable compactness (resp. the Lindelöf property) is a topological property.*

Proof Let $(X, \mathcal{T})$ be a countably compact (resp. Lindelöf) space, $(Y, \mathcal{S})$ a topological space, and $\Phi : X \to Y$ a homeomorphism. Since Φ is surjective, $Y = f(X)$, and by §1.9.71, Y is countably compact (resp. Lindelöf). ■

Proposition *2.9.73 Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{S})$ a countably compact (resp. Lindelöf) space, and $f : X \to Y$ a closed injection. Then $(X, \mathcal{T})$ is countably compact (resp. Lindelöf).*

Proof (Countably compact case) Let $\{U_i : i \in \mathbb{N}\}$ be an arbitrary countable open cover of X. Surely $\{f(U_i) : i \in \mathbb{N}\}$ is a countable open cover of $f(X)$, because f is necessarily open. Since f is closed, $(f(X))^c$ is open in Y, and $\{f(U_i) : i \in \mathbb{N}\} \cup \{(f(X))^c\}$ is a countable open cover of Y. By hypothesis, this cover admits a finite subcover. It follows that a finite subfamily of $\{U_i : i \in \mathbb{N}\}$ covers X, hence X is countably compact. The Lindelöf case goes through mutatis mutandi. ■

Corollary *2.9.74 Let $(X,\mathcal{T})$ be a countably compact (resp. Lindelöf) space, $A \subset X$, with A closed. Then A is countably compact (resp. Lindelöf).*

Proof The inclusion $inc : A \hookrightarrow X$ is closed by virtue of A being closed, and the appropriate case of §2.9.73 applies. ■

Countable compactness and the Lindelöf property are neither hereditary nor productive. Products which are known to be countably compact or Lindelöf pass the corresponding property to each factor space.

Example *2.9.75 We know from §2.9.39 that the ordinal spaces $[1,\omega]$ and $[1,\Omega]$ are compact, hence both countably compact and Lindelöf. But $[1,\omega)$ is not countably compact and $[1,\Omega)$ is not Lindelöf. To see the latter assertion, for example, consider the open cover $\{[1,\alpha) : \alpha \in \Omega\}$ of $[1,\Omega)$, which consists of an increasing nest of intervals. Now if there were a countable subcover, it would be of the form $\{[1,\alpha_n) : n \in \mathbb{N}\}$. But $\bigcup_{n\in\mathbb{N}}\{[1,\alpha_n)\} \subsetneq [1,\Omega)$, since the whole space is uncountable. Thus no countable subcover can exist.*

Example *2.9.76 The lower Sorgenfrey plane $\mathbb{R}^2_L$ (see §2.6.57) is a product of two Lindelöf factor spaces, which itself is not Lindelöf. The antidiagonal $\Delta^* = \{(x,y) : x = -y\}$ is closed in $\mathbb{R}^2_L$. To see this note that Δ^* cannot have any limit points in the half-plane $\{(x,y) : x < -y\}$ since the half-plane $\{(x,y) : x > -y\}$ is a neighborhood of Δ^*. And likewise, any point (ζ,ξ) in the half-plane $\{(x,y) : x > -y\}$ sits in a basic open rectangle with lower left corner at (ζ,ξ), and therefore cannot be a limit point of Δ^*. Hence $\mathbb{R}^2_L - \Delta^*$ is open. Consider the open cover of $\mathbb{R}^2_L$ consisting of $\mathbb{R}^2_L - \Delta^*$ and the open quarter-planes $\{(x,y) : x \geq \alpha, y \geq -\alpha\}$. Each point in Δ^* requires a distinct quarterplane to cover it, and since card$(\Delta^*) > \aleph_0$, $\mathbb{R}^2_L$ cannot be Lindelöf.*

Example *2.9.77 The* ***Novàk space*** *K is a subspace of $\beta\mathbb{N}$, the Stone-Čech compactification of the integers, which is countably compact, but for which $K \times K$ contains an infinite closed set discrete subset. Hence $K \times K$ cannot be countably compact.*

Both countable compactness and the Lindelöf property are divisible.

Proposition *2.9.78 Let $(X,\mathcal{T})$ be a countably compact (resp. Lindelöf) space and $R \subset X \times X$ an equivalence relation. Then X/R is countably compact (resp. Lindelöf).*

Proof The canonical map $q : X \to X/R$ is a continuous surjection, and both cases follow at once from §2.9.71. ■

As we have noted, the forerunner of modern compactness was sequential compactness.

Definition *2.9.79 Let $(X,\mathcal{T})$ be a topological space. If every sequence in X has a convergent subsequence, then we say X is* ***sequentially compact****. A subset of X is sequentially compact if it is sequentially compact in its own right as a subspace. In this case, the limits of the convergent subsequences must be contained in the subset.*

We note first that sequential compactness and compactness are independent, in general.

Example *2.9.80 The product space $[0,1]^{[0,1]}$ is compact by the Tychonoff Theorem, but not sequentially compact. Let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$.*

Now consider the sequence $(\chi_{q_n})_{n\in\mathbb{N}}$, consisting of the characteristic functions of these rationals. No subsequence of the characteristic functions can converge, so the space is not sequentially compact.

Example ***2.9.81*** *Now look at the subspace $\mathfrak{C}$ of $[0,1]^{[0,1]}$ consisting of functions with countable support. $\mathfrak{C}$ is sequentially compact, as we will show in §2.9.84, since the union of countably many countable supports is still countable. However it is not compact. If it were, it would be closed since $[0,1]^{[0,1]}$ is Hausdorff. Since $\mathfrak{C}$ contains the functions with finite support and rational range, and these are product-dense, it would follow that $\mathfrak{C} = cl\mathfrak{C} = [0,1]^{[0,1]}$, which is absurd.*

Sequential compactness is generally not hereditary, and not productive without restrictions, but it is preserved for closed subspaces, countable products, and arbitrary quotients.

Example ***2.9.82*** *The interval $[0,1]$ is sequentially compact, but its subspace $(0,1]$ is not, since the sequence $(n^{-1})_{n\in\mathbb{N}}$ and all of its subsequences can only converge to the limit zero, which has been stripped away.*

Example ***2.9.83*** *As noted, $[0,1]$ is sequentially compact, but its unrestricted product $[0,1]^{[0,1]}$ is not.*

Proposition ***2.9.84*** *Let $\{(X_n,\mathcal{T}_n) : n \in \mathbb{N}\}$ be a countable family of sequentially compact spaces. Then the product $\prod_{n\in\mathbb{N}} X_n$ is sequentially compact if and only if each factor space is sequentially compact.*

Proof (Necessity) Suppose X_n is sequentially compact for each $n \in \mathbb{N}$. Consider an arbitrary sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}$ in $X = \prod_{n\in\mathbb{N}} X_n$, where $\mathbf{x} = (x_1, x_2, \dots, x_n, \dots)$. We will adapt Cantor's diagonalization method to produce a convergent subsequence. Since X_1 is sequentially compact, we may choose a subsequence of $(\mathbf{x}_j)_{j\in\mathbb{N}}$ that converges in the first coördinate. Call this subsequence $(\mathbf{x}_j^{(1)})_{j\in\mathbb{N}}$. Now since X_2 is sequentially compact, we may choose a subsequence of $(\mathbf{x}_j^{(1)})_{j\in\mathbb{N}}$ that converges in the second coördinate. Call this subsequence $(\mathbf{x}_j^{(2)})_{j\in\mathbb{N}}$. Continuing recursively in this fashion, we choose $(\mathbf{x}_j^{(k)})_{j\in\mathbb{N}}$ to be a subsequence of $(\mathbf{x}_j^{(k-1)})_{j\in\mathbb{N}}$ that converges in the k^{th} coördinate. Now the sequence $(\mathbf{x}_j^{(j)})_{j\in\mathbb{N}}$ is a subsequence of every sequence $(\mathbf{x}_j^{(k)})_{j\in\mathbb{N}}$, and hence converges in every coördinate. But then $(\mathbf{x}_j^{(k)})_{j\in\mathbb{N}}$ is surely a convergent subsequence of the original arbitrary sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}$, and it follows that X is sequentially compact.

(Sufficiency) Let $X = \prod_{n\in\mathbb{N}} X_n$ be sequentially compact, and consider an arbitrary sequence $(x_j)_{j\in\mathbb{N}}$ in the k^{th} factor space. Choose any sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}$ in the product that has $(x_j)_{j\in\mathbb{N}}$ as its projection on the k^{th} coördinate. By hypothesis, $(\mathbf{x}_j)_{j\in\mathbb{N}}$ has a convergent subsequence, which in particular must converge in the k^{th} coördinate, and we conclude X_k is sequentially compact. Since k and $(x_j)_{j\in\mathbb{N}}$ were arbitrary, the result follows. ■

Proposition ***2.9.85*** *Let $(X,\mathcal{T})$ be a sequentially compact space, $(Y,\mathcal{S})$ a topological space, and $f : X \to Y$ a continuous function. Then $f(X)$ is sequentially compact.*

Proof Suppose $(y_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in Y. We will produce a convergent subsequence. For each $n \in \mathbb{N}$, $y_n = f(x_n)$ for some $x_n \in X$. By assumption, the sequence

$(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit, say x. But then since f is continuous, it converges along the elementary filter of $(x_{n_k}))_{k\in\mathbb{N}}$ to $f(x)$ by §2.5.7, i.e. $(y_{n_k}))_{k\in\mathbb{N}}$ converges to $y = f(x)$, and the result is immediate. ■

Proposition ***2.9.86*** *Let $(X, \mathcal{T})$ be a sequentially compact space, and $R \subset X \times X$ an equivalence relation. Then X/R is sequentially compact.*

Proof The canonical map $q : X \to X/R$ is a continuous surjection, and the result follows from applying §2.9.85. ■

Proposition ***2.9.87*** *Sequential compactness is a topological property.*

Proof Let $(X, \mathcal{T})$ be a sequentially compact space, $(Y, \mathcal{S})$ a topological space, and $\Phi : X \to Y$ a homeomorphism. Since Φ is a continuous surjection, the result follows from applying §2.9.85. ■

Proposition ***2.9.88*** *Let $(X, \mathcal{T})$ be a topological space, $(Y, \mathcal{S})$ a sequentially compact space, and $f : X \to Y$ a closed injection. Then $(X, \mathcal{T})$ is sequentially compact.*

Proof The map $f^{-1} : f(X) \to X$ is defined, since f is injective. Moreover f^{-1} is continuous by §2.5.8, since $(f^{-1})^{-1}$ takes closed sets into closed sets. Then by §2.9.85, $f^{-1}(f(X)) \ = X$ is sequentially compact. ■

Corollary ***2.9.89*** *Let $(X, \mathcal{T})$ be a sequentially compact space, $F \subset X$, with F closed. Then F is sequentially compact.*

Proof The inclusion map $inc : F \hookrightarrow X$ is a closed injection, and the result is immediate from §2.9.88. ■

There are several important connections that we can now establish among the varieties of compactness we have studied thus far.

Proposition ***2.9.90*** *Let $(X, \mathcal{T})$ be a sequentially compact space. Then $(X, \mathcal{T})$ is countably compact.*

Proof We may assume that X is infinite or there is nothing to prove. Suppose for the sake of contradiction that X is not countably compact. Then there exists a countable open cover $\{U_n : n \in \mathbb{N}\}$ of X which admits no finite subcover. Choose $x_1 \in U_1$, then recursively pick $x_n \in U_n - \bigcup_{i<n} U_i$. This is always feasible in view of the assumed property of the cover. Now $(x_n)_{n\in\mathbb{N}}$ is a sequence in X, and by hypothesis, there exists a convergent subsequence with limit, say x. But $x \in U_n$ for some n, and by construction, only one other point of the convergent subsequence could possibly be in U_n, which is absurd. It follows that X must be countably compact. ■

Lemma ***2.9.91*** *Let $(X, \mathcal{T})$ be a first countable topological space. Then each point $x \in X$ has a countable neighborhood base $\mathcal{B}_x = \{B_i : i \in \mathbb{N}\}$ which is a decreasing nest, i.e. $i \geq j$ implies $B_i \subset B_j$.*

Proof By hypothesis, an arbitrary point $x \in X$ has a countable base $\mathcal{B}'_x = \{B'_i : i \in \mathbb{N}\}$. Set $B_1 = B'_1$, $B_2 = B'_2 \cap B'_1$, and in general $B_n = \bigcap_{i=1}^n B'_i$. Clearly the collection $\{B_n : n \in \mathbb{N}\}$ consists of open neighborhoods of x which satisfy the nesting condition. ■

Proposition *2.9.92 Let $(X, \mathcal{T})$ be a first countable topological space. Then $(X, \mathcal{T})$ is sequentially compact if and only if it is countably compact.*

Proof (Necessity) Suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence in X, which, without restriction of generality, may be taken to consist of distinct terms. Claim: the infinite set $S = \{x_n : n \in \mathbb{N}\}$ has an ω-limit point. If not, then there must exist for each $x \in X$ an open neighborhood U_x such that $U_x \cap S$ is finite. Define $X_n = \bigcup_{i \le n} \{U_x : x_i \in U_x \cap S\}$. Now $\bigcup_{n \in \mathbb{N}} X_n = X$, and by countable compactness, there exists a finite index set H such that $\bigcup_{n \in H} X_n = X$. This is impossible, since the union contains only finitely many elements of S, and the claim therefore holds. So suppose x is an ω-limit point of S. By hypothesis, there exists a countable neighborhood base $\{B_k : k \in \mathbb{N}\}$ for $\mathcal{T}$ at x, which by §2.9.91 may be assumed to be a decreasing nest. Now we can choose $x_{n_1} \in B_1$, $x_{n_2} \in B_2 - \{x_{n_1}\}$, and in general $x_{n_k} \in B_k - \bigcup_{i<k} \{x_{n_i}\}$, because every neighborhood of x contains infintely many points of S. By construction the subsequence $(x_{n_k})_{k \in \mathbb{N}}$ converges to x, which gives the result.

(Sufficiency) §2.9.90. ■

Proposition *2.9.93 Let $(X, \mathcal{T})$ be a first countable T_1 topological space. Then the following are equivalent:*

(i) X **is limit point compact**
(ii) X **is countably compact**
(iii) X **is sequentially compact**

Proof (i) and (ii) are equivalent by §2.9.65, and (ii) and (iii) are equivalent by §2.9.92. ■

Corollary *2.9.94 Let $(X, \mathcal{T})$ be a second countable T_1 topological space. Then the following are equivalent:*

(i) X **is limit point compact**
(ii) X **is countably compact**
(iii) X **is sequentially compact**
(iv) X **is compact**

Proof Second countable spaces are a fortiori first countable, so §2.9.93 applies and handles the equivalence of (i), (ii), and (iii). §2.9.66 establishes the equivalence of (ii) and (iv). ■

Since the real numbers with their usual topology fit the condition of this corollary, early analysts had little incentive to abandon the familiarity of sequences in probing compactness phenomena. What we call compactness today was called bicompactness in the nineteenth century, but that term has been retired.

We continue with a discussion of some more modern ideas that relate to compactness. By and large, these arose as a byproduct of the so-called metrizability problem. This problem occupied general topology for the better part of the first half of the twentieth century. It asks which topological spaces are sufficiently structured so that a distance function can be developed which gives back the topology in a sense to be made precise in §3.1. We will take up this problem in detail in §3.2. First, we have a number of specialized notions regarding open covers. The index set A in the following definitions has no restriction.

Definition *2.9.95 Let $(X, \mathcal{T})$ be a topological space, and $\{U_\alpha : \alpha \in A\}$ be an open cover of X.*

Then if for each $x \in X$, there are at most finitely many indices α such that $x \in U_\alpha$, we say that the open cover is ***point-finite****.*

Definition *2.9.96 Let $(X, \mathcal{T})$ be a topological space, and $\{U_\alpha : \alpha \in A\}$ be an open cover of X. Then if for each $x \in X$, there is an open neighborhood N of x such that for at most finitely many indices $N \cap U_\alpha \neq \varnothing$, we say that the open cover is* ***neighborhood-finite*** *(****locally finite****). If for each $x \in X$, there is an open neighborhood N of x such that for only one index $N \cap U_\alpha \neq \varnothing$, we call the open cover* ***locally discrete****.*

Definition *2.9.97 Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X. The* ***star*** *of $\mathcal{U}$ with respect to a point $x \in X$ is the set $\mathcal{U}^* = \bigcup\{y \in U_\alpha : x \in U_\alpha \in \mathcal{U}\}$. If for each index α, $U_\alpha \cap U_\beta \neq \varnothing$ for at most finitely many indices $\beta \in A$, we say that the open cover $\mathcal{U}$ is* ***star-finite****. Equivalently, each $x \in X$ is contained in at most finitely many $U_\alpha \in \mathcal{U}$.*

Definition *2.9.98 Let X be a set and $\mathcal{U}, \mathcal{V} \in \wp(X)$. We say that $\mathcal{U}$ is a* ***refinement*** *of $\mathcal{V}$ (or $\mathcal{U}$* ***refines*** *$\mathcal{V}$, written $\mathcal{U} \ll \mathcal{V}$) if for every $U \in \mathcal{U}$ there exists some $V \in \mathcal{V}$ such that $U \subset V$. If X has a topology $\mathcal{T}$, then it is obviously possible to construct* ***open*** *or* ***closed refinements*** *consisting of the appropriate type of sets. In the event that $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ and $\mathcal{V} = \{V_\alpha : \alpha \in A\}$, with $U_\alpha \subset V_\alpha$ for each index α, we say that $\mathcal{U}$ is a* ***precise refinement*** *of $\mathcal{V}$.*

Let us be very clear that refinement in this special sense ($\ll$) is not the same as refinement in the case of one topology or filter refining another ($\prec$). It does not depend on one collection of sets being contained in the other as a subcollection, but rather hinges on the containment relations available *among the individual sets within the collections*. So $\mathcal{U} \ll \mathcal{V}$ does not mean that every set in $\mathcal{U}$ is also a set in $\mathcal{V}$, only that every set in $\mathcal{U}$ is contained in some set from $\mathcal{V}$. In particular, $\mathcal{U} \ll \mathcal{V}$ does not imply that $\mathcal{U}$ is a subcover of $\mathcal{V}$, although every subcover of a given cover is a trivial refinement of it. The sense of "$\ll$" is meant to remind us that the sets in $\mathcal{U}$ are "smaller" than the sets in $\mathcal{V}$ insofar as checking the defining property is concerned. Some texts write $\mathcal{U} \prec \mathcal{V}$ to mean that $\mathcal{V}$ refines $\mathcal{U}$ in this secondary sense, so awareness of the convention in force is advised.

Definition *2.9.99 Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a cover of X such that $\mathcal{U} - \{U_\alpha\}$ does not cover X for any α. Then $\mathcal{U}$ is called an* ***irreducible cover****.*

Point-finite and neighborhood-finite covers have an interesting minimal property that follows purely from set-theoretic considerations.

Proposition *2.9.100 Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a point-finite (resp. neighborhood-finite) cover of X. Then there exists an irreducible point-finite (resp. neighborhood-finite) cover $\mathcal{V}$ which is a subcover of $\mathcal{U}$.*

Proof (Point-finite case) We shall call a subfamily $\mathcal{W}$ of $\mathcal{U}$ superfluous if $\mathcal{U} - \mathcal{W}$ covers X. The collection of all superfluous families $\mathfrak{W}$ may be partially ordered by set inclusion. Suppose $\{\mathcal{W}_\alpha : \alpha \in A\}$ is a chain in $\mathfrak{W}$ for some index set A. Surely $\bigcup_{\alpha \in A} \mathcal{W}_\alpha$ is an upper bound for the chain. Claim: $\bigcup_{\alpha \in A} \mathcal{W}_\alpha$ belongs to $\mathfrak{W}$. If not, then $\mathcal{U} - \bigcup_{\alpha \in A} \mathcal{W}_\alpha$ does not cover X. But then for some $x \in X$, all of the finitely many $U_\alpha \in \mathcal{U}$ containing x do not belong to $\mathcal{U} - \bigcup_{\alpha \in A} \mathcal{W}_\alpha$. Now $\{\mathcal{W}_\alpha : \alpha \in A\}$ is a chain, hence all of these same U_α must not belong to one of the $\mathcal{W}_\alpha$, contradicting the superfluousness of that $\mathcal{W}_\alpha$.

Thus the claim is true, and since $\mathfrak{W}$ is inductively ordered, Zorn's Lemma ensures a maximal element, say $\mathcal{W}_\vee$. It follows that $\mathcal{V} = \mathcal{U} - \mathcal{W}_\vee$ is an irreducible point-finite open cover, as required. The neighborhood-finite case goes through mutatis mutandi. ■

Definition ***2.9.101*** *Let $(X, \mathcal{T})$ be a Hausdorff space. If every open cover of X admits a point-finite refinement, then $(X, \mathcal{T})$ is called **metacompact (pointwise paracompact)**.*

Proposition ***2.9.102 (Arens / Dugundji)*** *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is compact if and only if it is both countably compact and metacompact.*

Proof (Necessity) Let X be both countably compact and metacompact. Suppose $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an arbitrary open cover of X. By hypothesis, there exists a $\mathcal{V} \ll \mathcal{U}$, with $\mathcal{V}$ point-finite. By §2.9.100, $\mathcal{V}$ has an irreducible subcover, say $\mathcal{V}_\vee = \{V_\beta : \beta \in B\}$. We claim $\mathcal{V}_\vee$ is a finite subcover. Suppose not. Note that each V_β contains a point x_β, which is contained in no other member of $\mathcal{V}_\vee$ by irreducibility. By §2.9.65, X is limit point compact, hence the assumed infinite set $\{x_\beta : \beta \in B\}$ must have a limit point, say x. But $x \in V_{\beta_0}$ for some $\beta_0 \in B$, and evidently x cannot be an ω-limit point of $\{x_\beta : \beta \in B\}$, contrary to §2.8.7. The contradiction establishes our claim. Since $\mathcal{V}_\vee \ll \mathcal{U}$, we may find for each $V_\beta \in \mathcal{V}_\vee$ a $U_{\alpha(\beta)} \in \mathcal{U}$ such that $V_\beta \subset U_{\alpha(\beta)}$. Clearly $\mathcal{U}$ has a finite subcover $\{U_{\alpha(\beta)} : \beta \in B\}$, and hence X is compact.

(Sufficiency) By the definitions. ■

Definition ***2.9.103*** *Let $(X, \mathcal{T})$ be a Hausdorff space. If every open cover of X admits a neighborhood-finite refinement, then $(X, \mathcal{T})$ is called **paracompact**.*

Paracompact spaces were introduced by the French mathematician Jean Dieudonné in 1944 as a generalization of compact spaces. They moved toward center stage after 1951, when a complete and particularly natural characterization of metrizability in terms of paracompactness was offered by the Russian mathematician Y. M. Smirnov. Paracompact spaces exhibit some of the desirable features of compact Hausdorff spaces without being so restrictively defined. Of course, this has a price, and paracompactness fails to behave as agreeably as compactness itself under various common constructions and mappings.

Example ***2.9.104*** *Any compact Hausdorff space is paracompact. This is obvious once we restate compactness in terms of refinements of covers. A space is compact if every open cover admits a finite refinement. Surely finite refinements are neighborhood finite, and the statement follows.*

Example ***2.9.105*** *$\mathbb{R}^n$ with its usual topology or any infinite discrete space are paracompact, although neither is compact.*

Proposition ***2.9.106*** *Paracompactness is a topological property.*

Proof Let $(X, \mathcal{T})$ be a paracompact space, Let $(Y, \mathcal{S})$ a topological space, and $\Phi : X \to Y$ a homeomorphism. Suppose $\{V_\alpha : \alpha \in A\}$ is an arbitrary open cover of Y. Then $\{\Phi^{-1}(V_\alpha) : \alpha \in A\}$ is an open cover of X, which by assumption admits a neighborhood-finite refinement, say $\{U_\beta : \beta \in B\}$. But then $\{\Phi(U_\beta) : \beta \in B\}$ is a refinement of $\{V_\alpha : \alpha \in A\}$. For any point $y \in Y$, $\Phi^{-1}(y)$ has an open neighborhood N, which meets only finitely many of the sets U_β, and it follows that $\Phi(N)$ is an open neighborhood of y that meets only finitely many of the sets $\Phi(U_\beta)$. Therefore Y is paracompact. ■

Paracompactness is not invariant under continuous maps. This situation does not change even for continuous injections.

Example ***2.9.107*** *Consider the map* $id_{[1,\omega]} : [1,\omega] \to [1,\omega]$*, where the domain has the standard order topology, but the range has the topology with base* $\mathcal{B} = \{(n,\omega) : n \in \mathbb{N}\}$*. We know the domain is compact, hence paracompact, and* $id_{[1,\omega]}$ *is continuous, since the domain topology is finer than the range topology. But the range is not Hausdorff, and therefore cannot be paracompact.*

It has been established that paracompactness is preserved for closed continuous surjections, hence for perfect maps.

Proposition ***2.9.108 (Dieudonné)*** *Let* $(X,\mathcal{T})$ *be a paracompact space. Then* $(X,\mathcal{T})$ *is regular.*

Proof Let $Y \subset X$ be closed, and $x \notin Y$. By the Hausdorff property, for each $y \in Y$ there exist disjoint open sets U_y and V_y such that $y \in U_y$ and $x \in V_y$. The collection $\{U_y : y \in Y\} \cup (X - Y)$ is an open cover of X, which by paracompactness admits a neighborhood finite refinement, say $\{U_\alpha : \alpha \in A\}$. Define $U = \bigcup\{U_\alpha : U_\alpha \cap Y \neq \emptyset, \alpha \in A\}$. Now x must have an open neighborhood W that has nonvoid intersection with only finitely many of the sets U_α. Label these as $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}$. By the refinement condition, we can find sets U_{y_i} from the original collection such that $U_{y_i} \supset U_{\alpha_i}$ for $1 \leq i \leq n$. It follows that the corresponding sets V_{y_i} satisfy $V_{y_i} \cap U_{\alpha_i} = \emptyset$ for each i, and hence $V = W \cap (\bigcap_{i=1}^n V_{y_i})$ is disjoint from U. Since $x \in V$, $Y \subset U$, and $U, V \in \mathcal{T}$, we conclude that X is regular. ■

Proposition ***2.9.109 (Dieudonné)*** *Let* $(X,\mathcal{T})$ *be a paracompact space. Then* $(X,\mathcal{T})$ *is normal.*

Proof Let $Y, Z \subset X$ be disjoint closed sets. By the regularity established in §2.9.108, for each $y \in Y$ there exist disjoint open sets U_y and V_y such that $y \in U_y$ and $Z \subset V_y$. Again, the collection $\{U_y : y \in Y\} \cup (X - Y)$ is an open cover of X, which, by paracompactness admits a neighborhood-finite refinement, say $\{U_\alpha : \alpha \in A\}$. Let $U = \bigcup\{U_\alpha : U_\alpha \cap Y \neq \emptyset\}$. For each $z \in Z$, there is an open neighborhood W_z of z such that has nonvoid intersection with at most finitely many of the sets U_α. Label the latter for a particular z as $U^z_{\alpha_1}, \ldots, U^z_{\alpha_n}$. As in the preceding regularity argument, we can find sets $U^z_{y_i}$ from the original collection such that $U^z_{y_i} \supset U^z_{\alpha_i}$ for each $1 \leq i \leq n$. It follows that the corresponding sets $V^z_{y_i}$, defined in the obvious way through given pairings, satisfy $V^z_{y_i} \cap U^z_{\alpha_i} = \emptyset$ for each i, and $V^z = W_z \cap (\bigcap_{i=1}^n V^z_{y_i})$ is disjoint from U. Now set $V = \bigcup_{z \in Z} V^z$. All of the V^z were disjoint from U, hence $U \cap V = \emptyset$. Clearly, $Y \subset U$, $Z \subset V$, and $U, V \in \mathcal{T}$. It follows that X is normal. ■

The implication of the preceding proposition cannot generally be reversed.

Example ***2.9.110*** *We know from §2.7.67 that all well-ordered sets with the standard order topology are normal. Accordingly,* $[1,\Omega)$ *is normal. However, it is not paracompact, since the open cover* $\{[1,\alpha) : \alpha < \Omega\}$ *does not admit a neighborhood-finite refinement.*

Proposition ***2.9.111*** *Let* $(X,\mathcal{T})$ *be a paracompact space and* $F \subset X$*, with* F *closed. Then* F *is paracompact.*

Proof Suppose $\{U_\alpha : \alpha \in A\}$.is an open covering of F in the subspace topology. Since each U_α is of the form $V_\alpha \cap F$, where $V_\alpha \in \mathcal{T}$, we have $(\bigcup_{\alpha\in A} V_\alpha) \cup (X - F)$ as an open cover of X. By hypothesis, this cover has a neighborhood-finite refinement, say $\{W_\beta : \beta \in B\}$. But then the collection $\{W_\beta \cap F : \beta \in B\}$ is a refinement of $\{U_\alpha : \alpha \in A\}$ by sets open in the subspace topology, which is then clearly neighborhood-finite. ■

Example ***2.9.112*** *As with compactness, arbitrary subspaces of paracompact spaces need not be paracompact. The ordinal space $[1,\omega] \times [1,\Omega]$ is compact, hence paracompact by our earlier remarks. However, the subspace $[1,\omega] \times [1,\Omega] - \{(\omega,\Omega)\}$...recall this is the Tychonoff plank of §2.7.59...cannot be paracompact in view of §2.9.109, because it fails to be normal.*

Example ***2.9.113*** *Unlike the case for compactness, the product of paracompact spaces need not be paracompact. Denote by $\mathbb{R}_L$ the space $\mathbb{R}$ equipped with the lower Sorgenfrey topology (see §2.3.5). $\mathbb{R}_L$ is paracompact, but the lower Sorgenfrey plane $\mathbb{R}_L^2$ is not.*

Even if a space is not compact, it may display compactness in a neighborhood of each point. This is the notion of local compactness. As with paracompactness, we sacrifice some convenient properties of full compactness, but there is still much to be gained from the localized version. As usual, the reader is alerted to the fact that a given author may or may not invoke the Hausdorff condition as part of the definition of local compactness.

Definition ***2.9.114*** *Let $(X,\mathcal{T})$ be a topological space. If for $x \in X$ there exists an open neighborhood U of x such that clU is compact, then $(X,\mathcal{T})$ is said to be* ***locally compact at x****. If $(X,\mathcal{T})$ is locally compact at x for all $x \in X$, then it is called a* ***locally compact space****. A subset $Y \subset X$ is locally compact if it is locally compact as a subspace.*

Some texts refer to what we are calling local compactness as strong local compactness. The weaker form requires only that a point have a compact neighborhood. In Hausdorff spaces, strong and weak local compactness coalesce, since if each point $x \in X$ has a compact neighborhood N_x, then $intN_x$ is an open neighborhood with compact closure.

Definition ***2.9.115*** *Let $(X,\mathcal{T})$ be a topological space. If for every $x \in X$ there exists a base for $\mathcal{T}$ at x consisting of compact neighborhoods, then we say $(X,\mathcal{T})$ is a* ***compactly regular space****.*

Compact regularity is the property that we would *like* to call “local compactness”, since it would then be logically consistent with our pattern of defining other localized notions, however the accepted terminology is well established. We will verify shortly that for Hausdorff spaces, local compactness implies regularity, and the base of closed neighborhoods supplied by regularity (see §2.7.27) then gives us compact regularity via §2.9.13. Accordingly, we will confine ourselves to a study of local compactness after making only a few basic observations about compact regularity.

Example ***2.9.116*** *$\mathbb{R}^n$ is locally compact and compactly regular for all n. $\mathbb{R}^{\mathbb{N}}$ is neither. The spaces $[1,\omega)$ and $[1,\Omega)$ are both noncompact, but locally compact and compactly regular.*

Unlike compactness, local compactness is not invariant under continuous maps. However, for so-called **interior maps**, which are both open and continuous, local compactness is preserved.

Example *2.9.117 Let $X = \{x \in \mathbb{R} : x = -1$ or $0 < x < 1\}$ and $Y = \{(x,y) \in [0,1]^2 : y = 0$ if $x = 0$, and $y = \sin\frac{1}{x}$ otherwise$\}$. Endow both spaces with their natural subspace topologies. Now consider $f : X \to Y$ such that $f(-1) = (0,0)$ and $f(x) = (x, \sin\frac{1}{x})$ if $0 < x < 1$. X is locally compact and f is surely continuous by construction. But we claim that Y is not locally compact. Y is regular as the subspace of the regular space $\mathbb{R}^2$. If it were locally compact, then every open set containing the origin would contain a compact subset. But this is impossible in view of the fact that every such set contains an unrectifiable part of the graph of $y = \sin\frac{1}{x}$, and we can arrange to cover this infinitely long curve with an infinite irreducible open cover.*

Proposition *2.9.118 Let $(X, \mathcal{T})$ be a locally compact (resp. compactly regular) topological space, $(Y, \mathcal{S})$ topological space, and $f : X \to Y$ an open continuous map. Then $f(X)$ is locally compact (resp. compactly regular).*

Proof (Locally compact case) Given $y \in f(X)$ find $x \in X$ such that $y = f(x)$. By assumption, x has a neighborhood U such that clU is compact. Since f is open, $f(U)$ must be an open neighborhood of y. Now $f(clU)$ is compact by §2.9.11, and since $int(f(clU)) \supset f(U) \ni y$, the result is immediate. The compactly regular case goes through similarly, mutatis mutandi. ■

Proposition *2.9.119 Local compactness (resp. compact regularity) is a topological property.*

Proof (Both cases) Let $(X, \mathcal{T})$ be a locally compact (resp. compactly regular) space, $(Y, \mathcal{S})$ topological space, and $\Phi : X \to Y$ a homeomorphism. $Y = \Phi(X)$, and §2.9.118 applies, establishing the result for both cases. ■

Proposition *2.9.120 Let $(X, \mathcal{T})$ be a locally compact space and $Y \subset X$, with Y open (resp. closed). Then Y is locally compact.*

Proof (Open case) Since Y is open in X, every subspace open set is open in $\mathcal{T}$. It follows that $inc_Y : Y \to X$ is a surjective interior map, hence by §2.9.118, Y is locally compact.

(Closed case) Suppose Y is closed in X and $y \in Y$. As a point in X, y has an open neighborhood U whose $\mathcal{T}$-closure is compact. Then $U \cap Y$ is a subspace open neighborhood of y. Now $cl_{\mathcal{T}}(U \cap Y)$ is compact by §2.9.13, and clearly $cl_{\mathcal{T}|Y}(U \cap Y) = cl_{\mathcal{T}}(U \cap Y) \subset cl_{\mathcal{T}}(Y) = Y$. It follows that Y is locally compact. ■

We have available a Tychonoff type theorem for locally compact spaces, but it is necessary for almost all of the factor spaces to be fully compact.

Proposition *2.9.121 Let $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in A\}$ be a family of locally compact spaces for some arbitrary index set A. Then $X = \prod_{\alpha \in A} X_\alpha$ is locally compact if and only if X_α is locally compact for at most finitely many α and compact for all other indices.*

Proof (Necessity) Suppose X_α is locally compact but not compact for α_i, $i \in H$, a finite index set, and compact for all $\alpha \in A - H$. Given an arbitrary point $\mathbf{x} = (x_\alpha) \in X$, we need to produce a compact neighborhood of $\mathbf{x}$. By hypothesis, for each $i \in H$ there exists a compact neighborhood $U_{\alpha_i} \ni x_{\alpha_i}$. It follows that $\left(\prod_{i \in H} U_{\alpha_i}\right) \times \left(\prod_{\alpha \in A-H} X_\alpha\right)$ is both a product neighborhood of $\mathbf{x}$, and a compact set by the Tychonoff Theorem. This establishes necessity.

(Sufficiency) Suppose the product is locally compact. Coördinate projections are open and continuous surjections, hence by §2.9.118, each factor space must be locally compact. Moreover, at most finitely many factor spaces may be noncompact. Suppose to the contrary that we have infinitely many locally compact/noncompact factors in the product. Consider an arbitrary compact product neighborhood. All but finitely many factors in the compact neighborhood must be full spaces, so this neighborhood must have noncompact factors. But then by Tychonoff's Theorem, it cannot be compact. The contradiction establishes sufficiency. ■

The limitation on the number of noncompact factors in this "local Tychonoff" result is due to the definition of the product topology. If we try to remove the restriction and form an infinite product of compact neighborhoods which are not whole factor spaces, we pass out of the realm of the product topology and into that of the box topology.

A number of further basic results are available if we agree that our locally compact spaces also be Hausdorff.

Proposition ***2.9.122*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space. Then $(X, \mathcal{T})$ is regular.*

Proof Suppose $x \in U \in \mathcal{T}$ is arbitrary. By hypothesis, there exists an open neighborhood V of x such that clV is compact. Now $U \cap V$ is an open neighborhood in both $\mathcal{T}$ and its relativization to clV. By §2.9.30, clV is a regular space, so there must exist some subspace-closed neighborhood W of x such that $W \subset U \cap V$. Then by §2.9.20, clV is seen to be $\mathcal{T}$-closed. Now we have $x \in W \subset U$, with W closed, which establishes regularity by §2.7.27. ■

Proposition ***2.9.123*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space and $K \subset X$ compact. Then every neighborhood of K contains a compact neighborhood.*

Proof Suppose $U \supset K$, where $U \in \mathcal{T}$. By §2.9.122 and our prefatory remarks, X is compactly regula, i.e. each $x \in X$ has a base of compact neighborhoods. For each $x \in K$, let C_x be a compact neighborhood of x contained in U. Then $\{intC_x : x \in K\}$ is an open cover of K, which by assumption is reducible to a finite subcover, say $\{intC_{x_i} : 1 \leq i \leq n\}$. But then $\bigcup_{i \leq n} C_{x_i}$, as a finite union of compact sets, is itself compact, contains K, and is contained in U, as required. ■

Proposition ***2.9.124*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space and $A \subset X$. Then A is closed if and only if for all compact sets $K \subset X$, $K \cap A$ is closed.*

Proof (Necessity) Suppose for the sake of contradiction that A is not closed. Then there exists a point $x \in clA - A$. By hypothesis, x has an open neighborhood U such that clU is compact. By assumption, $clU \cap A$ is closed, hence $clU \cap A = cl(clU \cap A) = clU \cap clA$. Noting that $x \in clU \cap clA$ and therefore $x \in clU \cap A$. But this means $x \in A$, contrary to supposition. It follows that A must be closed.

(Sufficiency) Suppose A is closed. By §2.9.20, K is closed, hence $K \cap A$ is closed. ■

To paraphrase the last result, under the given conditions a set is closed if and only if it is **locally closed**. Local closure is tested by the base of compact, hence closed, sets available.

Proposition ***2.9.125*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space and $Y \subset X$. Then Y is locally compact if and only if Y is representable as $F \cap G$, where F is closed and G is* *open.*

Proof (Necessity) By §2.9.120, both F and G are locally compact. Then for $x \in Y = F \cap G$, there exist open neighborhoods $U \subset F$ and $V \subset G$ such that $x \in U \cap V$ with both clU and clV compact. Then $clU \cap clV = cl(U \cap V)$ is compact, showing Y to be locally compact.

(Sufficiency) Suppose Y is locally compact. We may assume for each $y \in Y$ there exists a neighborhood of the form $V_y \cap Y$, which is compact for the relativization of $\mathcal{T}$ to Y, and where V_y is a $\mathcal{T}$-neighborhood. Then $V_y \cap Y$ is compact in X, and by §2.9.20, $\mathcal{T}$-closed as well. Now $int_{\mathcal{T}}(V_y \cap Y) = G_y \cap Y$, for some $G_y \in \mathcal{T}$. Accordingly, we have $G_y \cap cl_{\mathcal{T}}Y \subset cl_{\mathcal{T}}(G_y \cap Y) = cl_{\mathcal{T}}(int_{\mathcal{T}}(V_y \cap Y)) \subset V_y \cap Y$. It follows that $Y = \bigcup_{y \in Y}(G_y \cap cl_{\mathcal{T}}Y)$, which is an open set in the subspace $cl_{\mathcal{T}}Y$. Apparently $Y = U \cap cl_{\mathcal{T}}Y$, where $U \in \mathcal{T}$, and this establishes the representation. ■

Results for quotients of locally compact spaces are weaker than those for compact spaces. Recall that a compact equivalence relation R on X is one in which $K \subset X$ and K compact imply $sat_R(K)$ compact.

Proposition ***2.9.126*** *Let $(X, \mathcal{T})$ be a locally compact Hausdorff space, $R \subset X \times X$ a compact equivalence relation, and X/R a Hausdorff space. Then R is a closed equivalence relation, X/R is locally compact, and the canonical map $q : X \to X/R$ is a compact map.*

Proof First we must show that if $F \subset X$ is closed, then $sat_R(F)$ is also closed. Suppose not. Then there exists some $x \in clsat_R(F) - sat_R(F)$. Note that $q(x)$ must be a limit point of $q(F)$, otherwise there would exist an open neighborhood U of $q(x)$ such that $U \cap q(F) = \varnothing$. Then $q^{-1}(U)$ would contain x and $q^{-1}(U) \cap sat_R(F) = \varnothing$, contrary to assumption, establishing that R is a closed relation. Suppose now that $U \subset X/R$ is an open neighborhood of y. Then $q^{-1}(U)$ is open, and by assumption contains a compact neighborhood K of the fiber $q^{-1}(y)$. It follws from §2.9.11 that $q(K)$ is a compact neighborhood of y, and since y was arbitrart, X/R must be locally compact. Finally, suppose $C \subset X/R$ is compact, hence closed by §2.9.20. For each $x \in C$, let V_y be a compact neighborhood of some $x \in X$ such that $y = q(x)$. Clearly $q(V_y)$ is a compact neighborhood of y, and in view of the compactness of C, there is a finite index set H such that $\{q(V_{y_i}) : i \in H\}$ covers C. Now $sat_q(V_{y_i})$ is compact by hypothesis, and evidently $\bigcup_{i \in H} sat_q(V_{y_i}) \supset q^{-1}(C)$. But by continuity, $q^{-1}(C)$ is closed, hence compact by §2.9.13. ■

Local compactness and paracompactness are related, but first we need a concept that describes how a noncompact, but locally compact, space might be built up out of compact subspaces.

Definition ***2.9.127*** *Let $(X, \mathcal{T})$ be a locally compact space. If $(X, \mathcal{T})$ can be represented as the countable union of compact spaces, then we say $(X, \mathcal{T})$ is* ***σ-compact (countable at infinity)****.*

This definition fits the usual convention. If a set has property **P**, then a countable union of such sets is **σ-P**.

Example ***2.9.128*** *$\mathbb{R}$ can be written as $\bigcup_{n \in \mathbb{N}}[-n, n]$, where each interval $[-n, n]$ is compact by the Heine-Borel Theorem. Since each $x \in \mathbb{R}$ must be in the interior of an interval of this form, $\mathbb{R}$ is locally compact, and by the countability of the union, it is also σ-compact.*

Proposition ***2.9.129*** *Let $(X,\mathcal{T})$ be a locally compact Hausdorff space. If $(X,\mathcal{T})$ is σ-compact, then it is paracompact.*

Proof Suppose $X = \bigcup_{n\in\mathbb{N}} X_n$, where each X_n is compact. We can assume that the family $\{X_n : n \in \mathbb{N}\}$ forms an increasing nest...otherwise set $X'_n = \bigcup_{i\leq n} X_i$ and observe that the family $\{X'_n : n \in \mathbb{N}\}$ has the required structure. Claim #1: $X = \bigcup_{n\in\mathbb{N}} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a family of open sets such that clU_n is compact and $clU_n \subset U_{n+1}$. By local compactness, each point $x \in X_1$ has an open neighborhood V_x such that clV_x is compact. The family $\{V_x : x \in X_1\}$ is an open cover of X_1, hence there exists a finite subcover, say $\{V_{x_k} : k \in H_1\}$. Set $U_0 = \varnothing$ and $U_1 = \bigcup_{k\in H_1} V_{x_k}$. Clearly $clU_0 \subset U_1$ and clU_1 is compact. Repeating the construction, consider $W_2 = clU_1 \cup X_2$. Again by local compactness, each $w \in W_2$ has an open neighborhood V_w with compact closure, and since W_2 is compact, we can reduce the open cover $\{V_w : w \in W_2\}$ to a finite subcover $\{V_{w_k} : k \in H_2\}$. Then if we set $U_2 = \bigcup_{k\in H_2} V_{w_k}$, we have $clU_1 \subset U_2$ with clU_2 compact. Continuing in this way, we obtain a sequence of open sets $\{U_n : n \in \mathbb{N}\}$ such that clU_n is compact and $clU_n \subset U_{n+1}$. Moreover, since $X_n \subset U_n$, we have $\bigcup_{n\in\mathbb{N}} X_n \subset \bigcup_{n\in\mathbb{N}} U_n \subset X$, and the first claim is established.

Consider an arbitrary open cover $\mathcal{W} = \{W_\alpha : \alpha \in A\}$ of X. Let $K_n = clU_n \cap U^c_{n-1}$ and note that $\bigcup_{n\in\mathbb{N}} K_n = X$. Each K_n is closed by §2.9.20 and compact by §2.9.13. Claim #2: For each $x \in K_n$ and each $W_\alpha \ni x$, there exists an open set G_x with the following properties: (i) $G_x \subset W_\alpha$, (ii) $G_x \subset U_{n+1}$, and (iii) $G_x \cap clU_{n-2} = \varnothing$. Consider the set $G_x = W_\alpha \cap U_{n+1} \cap (clU_{n-2})^c$. This is an open neighborhood of x because $K_n \subset U_{n+1}$ and since $clU_{n-2} \subset U_{n-1}$, we have $K_n \cap clU_{n-2} \subset K_n \cap U_{n-1} = \varnothing$. This establishes the second claim.

Now for a given K_n, the collection $\{G_x : x \in K_n\}$ is an open cover, and by compactness, there exists a finite subcover, call it $\mathcal{G}_n$. We assert that $G = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ is a neighborhood-finite refinement of $\mathcal{W}$. Surely $\mathcal{G} \ll \mathcal{W}$ by construction. Given $x \in X$, we may find $j = \inf\{n \in \mathbb{N} : x \in clU_n\}$. Then $x \notin K_l$ for $l \leq j-1$. Also, since $x \in clU_j \subset U_{j+1}$, $x \notin K_l$ for $l \geq j+2$. Evidently, either $x \in K_l$ or $x \in K_{l+1}$. By construction (the sets in $\mathcal{G}$ have the properties (i) and (ii) above), there exists a $G \in \mathcal{G}$ such that $x \in G$ and $G \cap H = \varnothing$ if $H \in \mathcal{G} - (\mathcal{G}_{j-2} \cup \mathcal{G}_{j-1} \cup \mathcal{G}_j \cup \mathcal{G}_{j+1})$. But $\mathcal{G}_n$ is finite for each index n, hence $\mathcal{G}$ is locally finite, and we conclude that X is paracompact. ■

The cover which was constructed in the preceding proposition is also star-finite.

Proposition ***2.9.130*** *Let $(X,\mathcal{T})$ be a locally compact, second countable Hausdorff space. Then $(X,\mathcal{T})$ is paracompact.*

Proof If $\{B_n : n \in \mathbb{N}\}$ is a countable global base for $\mathcal{T}$, we can find a countable base $\{C_n : n \in \mathbb{N}\}$ consisting of compact neighborhoods by §2.9.122. But then $X = \bigcup_{n\in\mathbb{N}} C_n$, and by §2.9.129, X is paracompact. ■

This section continues with a discussion of compactification, which is a process for manufacturing a compact space from one that is not. Our discussion is limited in the sense that we do not have available various notions from the theory of uniform spaces that would support a more comprehensive treatment. This will not obscure the usefulness of the idea, nor prevent us from obtaining Alexandroff's groundbreaking result on the "one-point" compactification.

Definition ***2.9.131*** *Let $(X,\mathcal{T})$ be a topological space, and $(Y,\mathcal{S})$ a compact space. If $\kappa : X \to Y$*

is a topological embedding and $\kappa(X)$ is dense in Y, then the pair $(\kappa, (Y, \mathcal{S}))$ is called a ***compactification of* X**. *If $(Y, \mathcal{S})$ is Hausdorff, then $(\kappa, (Y, \mathcal{S}))$ is called a* ***Hausdorff compactificatio****n.*

We will routinely abuse the terminology and say that Y is a compactification of X. It is apparent at once that compactifiable spaces must be homeomorphic to subspaces of compact spaces, and therefore possess all of the corresponding hereditary properties. An immediate conclusion is that compactifiable Hausdorff spaces must be completely regular. The purpose of building the denseness condition into the definition is to economize on the codomain and rule out unnecessarily extravagant compactifications.

Any topological space admits a particularly simple compactification - it amounts to adding a single "ideal point" to the original space and cleverly expanding the original topology to embrace the new point in a way that makes the whole thing compact. Paul Alexandroff introduced this method in 1929. The Alexandroff compactification is necessarily the smallest that can be engineered for a suitable space. In general, there are many compactifications possible for a given space, and these can be partially ordered by set inclusion among the codomains. The largest possible compactification in this scheme is the Stone-Čech compactification, and all other compactifications can be viewed as various quotients of it.

Proposition ***2.9.132 (Alexandroff)*** *Let $(X, \mathcal{T})$ be a topological space. Then there exists a compactification of X, denoted by $(\kappa(X_\infty, \mathcal{T}_\infty))$, such that $X_\infty - \kappa(X)$ consists of one point. Moreover, $(X_\infty, \mathcal{T}_\infty)$ is a Hausdorff compactification if and only if $(X, \mathcal{T})$ is a locally compact Hausdorff space. In this instance, the space $(X_\infty, \mathcal{T}_\infty)$ is unique up to homeomorphism.*

Proof (Existence) Note that the space X_∞ is called the **one-point** (or **Alexandroff**) compactification. This symbol should not be confused with the notation for direct limit space in §2.6. Let us use the symbol "∞" to mean a point not in X. We topologize $X_\infty = X \cup \{\infty\}$ as follows. Denote by $\mathcal{T}'$ the collection $\{V \subset X_\infty : V = K^c, K$ $\mathcal{T}$-compact$\}$. Sets in this collection are complements, taken in the augmented space, of compact sets in the original space. Both $\mathcal{T}$ and $\mathcal{T}'$ are separately stable under finite intersections, and for every $F \in \mathcal{T}$ and $G \in \mathcal{T}'$, $F \cap G \in \mathcal{T}$. Evidently $\mathcal{T} \cap \mathcal{T}'$ is a base for a topology $\mathcal{T}_\infty$ on X_∞,and $\mathcal{T}_\infty|_X = \mathcal{T}$. X_∞ is compact for $\mathcal{T}_\infty$, for suppose $\{U_\alpha : \alpha \in A\}$ is an arbitrary open cover of X_∞. Some set $U_{\alpha(\infty)}$ has to cover ∞. The complementary space $X_\infty - U_{\alpha(\infty)}$ is compact by construction, hence it admits a finite subcover. Adjoin $U_{\alpha(\infty)}$ to this finite subcover, and we have a finite subcover of X_∞. Thus $(X_\infty, \mathcal{T}_\infty)$ is a compact space, and ∞ is a limit point of X in this setup, so X is dense in X_∞. Let $\kappa = inc_X : X \to X_\infty$, and we have the compactification $(inc_X, (X_\infty, \mathcal{T}_\infty))$.

(Hausdorff necessity) To show that $\mathcal{T}_\infty$ is Hausdorff whenever $\mathcal{T}$ is Hausdorff and locally compact, it is clearly enough to consider separating an arbitrary $x \in X$ from ∞. By local compactness, x has an open neighborhood U such that clU is compact. Then $x \in U$, $\infty \in (clU)^c$, which is open by construction, and $U \cap (clU)^c = \emptyset$, from which it follows that $\mathcal{T}_\infty$ is Hausdorff.

(Hausdorff sufficiency) Conversely, if $\mathcal{T}_\infty$ is Hausdorff, its restriction to X must be Hausdorff as well by §2.7.14. But this is just $\mathcal{T}$. Now for any $x \in X$, by assumption there are sets U and V such that $x \in U$, $\infty \in V$, and $U \cap V = \emptyset$. V^c is compact by construction, and evidently $x \in U \subset int(V^c)$, which establishes local compactness.

(Hausdorff uniqueness) By relabeling the "ideal point", we can suppose that any other one-point compactification results in the compact space $(X_\infty, \mathcal{T}'_\infty)$, where $\mathcal{T}'_\infty|_X = \mathcal{T}$. Claim: the $\mathcal{T}_\infty$-neighborhood filter $\mathcal{N}_\infty$ must $\mathcal{T}'_\infty$-converge to ∞. By compactness, adh$\mathcal{N}_\infty \neq \emptyset$, and by the Hausdorff condition, no $x \in X$ can be contained in $adh\mathcal{N}_\infty$. Then by §2.9.18, $\mathcal{N}_\infty \to \infty$ for $\mathcal{T}'_\infty$, establishing the claim. Evidently $\mathcal{N}_\infty$ refines the $\mathcal{T}'_\infty$-neighborhood filter $\mathcal{N}'_\infty$. Reversing the roles of $\mathcal{N}_\infty$ and $\mathcal{N}'_\infty$, we conclude that $\mathcal{N}_\infty = \mathcal{N}'_\infty$. We see that the fundamental systems of neighborhoods at each point in X_∞ are the same for both topologies, hence the topologies agree. It follows that id_{X_∞} is a homeomorphism, establishing the essential uniqueness of the compactification. ■

Example ***2.9.133*** $(\mathbb{R}_{0+})_\infty = [0, \infty]$, $\mathbb{R}_\infty \cong S_1$, *the circle, and* $\mathbb{R}^2_\infty \cong S_2$, *the sphere, where the homeomorphism is a stereographic projection. The Banach space c of convergent real sequences can be viewed as* $C(\mathbb{N}_\infty)$, *the space of continuous real functions on the Alexandroff compactification of the integers.*

Example ***2.9.134*** *The uniqueness part of §1.9.132 allows us to conclude that the Alexandroff compactification of* $(0,1]$ *is* $[0,1]$, *and that of* $[1,\omega)$ *is* $[1,\omega]$. *Also, since stereographic projection is a homeomorphism between the complex plane and the Riemann sphere minus its north pole, evidently the entire sphere is the Alexandroff compactification of the complex plane.*

Proposition ***2.9.135*** *Let* $(X, \mathcal{T})$ *be a locally compact Hausdorff space. Then* $(X, \mathcal{T})$ *is a Tychonoff space.*

Proof If X is compact, then it is necessarily T_4 by §2.9.31, hence a Tychonoff space by §2.7.73. If X is not compact, then it has an Alexandroff compactification X_∞ by §2.9.132. The first statement then applies to the compact Hausdorff space X_∞, and X is a Tychonoff space by §2.7.43. ■

We conclude this section with a brief characterization of a class of spaces that includes the locally compact Hausdorff spaces. The topology of these spaces is completely determined by the available closed, compact sets.

Definition ***2.9.136*** *Let* $(X, \mathcal{T})$ *be a topological space and* $\mathcal{K}$ *denote the family of all closed, compact subsets of X. Then* $(X, \mathcal{T})$ *is called a* **k-*space***, *or said to be* ***compactly generated***, *if the following condition is true. For any* $Y \subset X$, *if* $Y \cap K$ *is closed for every* $K \in \mathcal{K}$, *then Y is closed.*

If Y is closed, then $Y \cap K$ is closed trivially, hence for k-spaces, a set Y is closed if and only if $Y \cap K$ is closed for every $K \in \mathcal{K}$. Evidently the closed sets, hence the topology, of a k-space are specified completely by the preceding condition. In a compactly generated space, if U is relatively open in $K \in \mathcal{K}$, then U^c is relatively closed in K, hence closed in X. It follows that $U \in \mathcal{T}$. The converse is trivial, so in a compactly generated space, a set is open if and only if its intersection with every $K \in \mathcal{K}$ is relatively open in K.

Proposition ***2.9.137*** *Let* $(X, \mathcal{T})$ *be a locally compact Hausdorff space or a first countable Hausdorff space. Then* $(X, \mathcal{T})$ *is a k-space.*

Proof (Locally compact case) We will show the contrapositive, namely that for an arbitrary non-closed $Y \subset X$, there must be some closed compact $K \subset X$ such that $Y \cap K$ is not closed. By assumption, there exists some $y \in clY - Y$. In the locally compact case, we have a closed compact neighborhood C of y. Now y is contained in C and Y accumulates at y. Then $C \cap Y$ cannot be closed, since y is alimit point of $C \cap Y$ but $y \notin C \cap Y$. This takes care of the locally compact case.

(First countable case) For the first countable case, we can arrange for a nonstationary sequence $(y_n)_{n \in \mathbb{N}}$ to converge to a unique y by §2.8.21. Since the union of a sequence with its limit is compact, $K = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ is a closed compact set which intersects Y in the non-closed set $\{y_n : n \in \mathbb{N}\}$. This establishes the first countable case. ■

We will add to our compactness results in §3.0, when some fundamental uniform notions become available in connection with metric spaces.