Rook Polynomials

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1 Introduction

In chess, a rook is able to capture pieces in the same row or column as the rook. We take this idea and apply it to combinatorical problems which involve permutations with forbidden positions. By using generating functions, and nxm arrays, we construct rook polynomials. These polynomials simplify certain problems in combinatorics that, otherwise, would be much more difficult to solve. In this paper we will cover the basic construction of rook polynomials, calculus of non-disjoint boards, and derangements.

2 Rook Polynomials

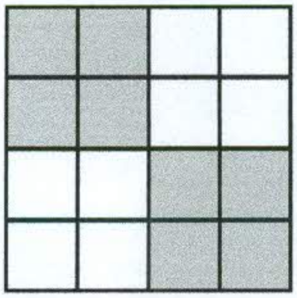
Definition: A board is an nxm array of n rows, and m columns. When a board has a darkened square, it is said to have a forbidden position.

Definition: A rook polynomial on a board B, with forbidden positions is denoted as $R(x, B)$.

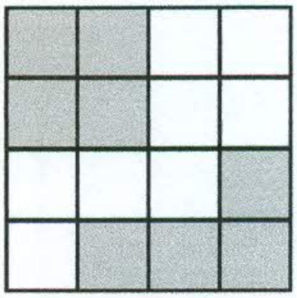
$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + ...r_k(B)x^k$$

The coefficients $r_k(B)$'s of $R(x, B)$ represent the number of noncapturing rooks on board B. Since there is only one way to place no rooks, we define $r_0(B) = 1$.

Definition: A board B, with forbidden positions, is said to be disjoint if the board can be decomposed into two subboards B_1 and B_2 such that, neither B_1 nor B_2 share the same row or column.



(a)



(b)

In the figures above, (a) is disjoint; however, (b) is not. This is because the square in row 4 shares the same column as row 1 and 2.

Boards are invariant, they may be rearranged by swapping rows with other rows, and by swapping columns with other columns. This allows us to attempt to make nondisjoint boards into disjoint boards.

Theorem 1: If B is a board of darkened squares that decomposes into two disjoint subboards B_1 and B_2 then, $R(x,B) = R(x,B_1)R(x,B_2)$.

Proof. Let

$$R(x,B_1) = \sum_{k=0}^{n} r_k(B_1)x^k = 1 + r_1(B_1)x + r_2(B_1)x^2 + ...r_n(B_1)x^n$$

and

$$R(x,B_2) = \sum_{k=0}^{n} r_k(B_2)x^k = 1 + r_1(B_2)x + r_2(B_2)x^2 + ...r_n(B_2)x^n$$

Then, $R(x, B_1)R(x, B_2) = \sum_{k=0}^{n} r_k(B_1)x^k \sum_{k=0}^{n} r_k(B_2)x^k$. The nth coefficient of $R(x, B_1)R(x, B_2)$ is

$$r_0(B_1)r_n(B_2) + r_1(B_1)r_{n-1}(B_2) + ...r_n(B_1)r_0(B_2)$$

This demonstrates that when there is no rook on B_1 there are n rooks on B_2. This continues on with 1 rook on B_1, $n-1$ on B_2; all the way to n rooks on B_1 and no rooks on B_2. Thus,

$$R(x, B_1)R(x, B_2) = \sum_{k=0}^{n} r_k(B_1)x^k \sum_{k=0}^{n} r_k(B_2)x^k = \sum_{k=0}^{n} r_k(B)x^k = R(x, B)$$

□

The following theorem is from the Inclusion-Exclusion Principle, it allows us to count the number of arrangements of distinct objects using rook polynomials.

Theorem 2: The number of ways to arrange n objects among m positions ($m \geq n$) when there are restricted positions is [Michaels]:

$$P(m, n) - r_1(B)P(m-1, n-1) + r_2(B)P(m-2, n-2) + ...(-1)^n r_n(B)P(m-n, 0)$$

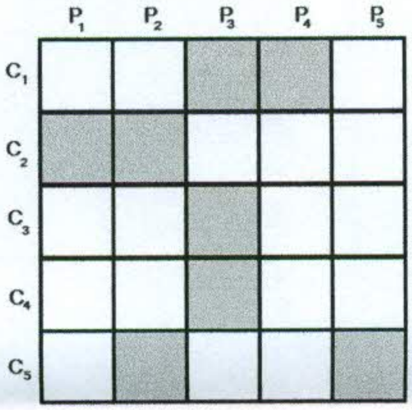
When $m = n$,

$$n! - r_1(B)(n-1)! + r_2(B)(n-2)! + ...(-1)^n r_n(B)0!$$

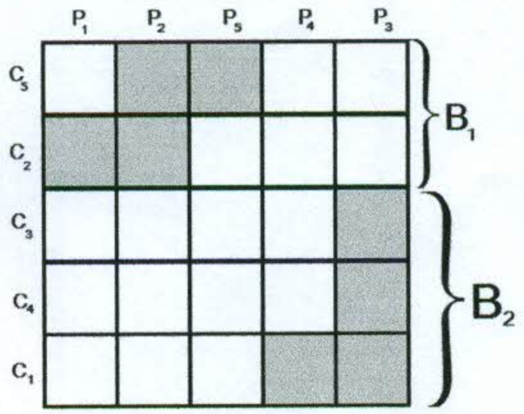
$$= \sum_{k=0}^{n} (-1)^k r_k(B)(n-k)!$$

Now we show a basic example of how rook polynomials are used to solve counting problems with forbidden positions.

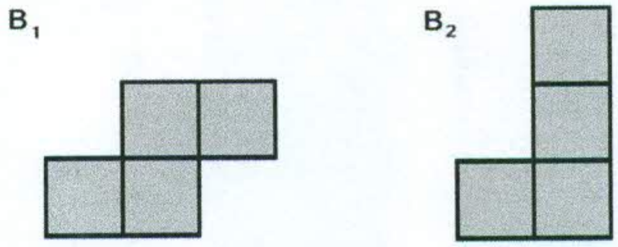
Example 1: How many ways are there to assign five different classes, denoted C_1, C_2, C_3, C_4, C_5, to five professors denoted as P_1, P_2, P_3, P_4, P_5, if P_1 would not like to teach C_2; if P_2 would not like to teach C_2 and C_5; if P_3 would not like to teach C_1, C_3, and C_4; if P_4 would not like to teach C_1; and if P_5 would not like to teach C_5?
Solution: We begin by constructing a 5x5 board, B, with darkened squares for the given forbidden positions.



Next, we attempt to rearrange B into two disjoint subboards B_1 and B_2. We find the complete, rearranged board to be:



Notice row C_1 was swapped with C_5; column P_5 was swapped with P_3. The board B is now decomposed into two disjoint subboards B_1 and B_2, because neither subboard shares the same column or row with the other.



By definition, we know $r_0(B_1) = r_0(B_2) = 1$. Then, by inspection, we find $r_1(B_1) = r_1(B_2) = 4; r_2(B_1) = 3, r_2(B_2) = 2$. The rook polynomial of B_1 is

$$R(x, B_1) = \sum_{k=0}^{2} R_k(B_1)x^k = r_0(B_1)x^0 + r_1(B_1)x^1 + r_2(B_1)x^2 = 1 + 4x + 3x^2$$

The rook polynomial of B_2 is

$$R(x, B_2) = \sum_{k=0}^{2} R_k(B_2)x^k = r_0(B_2)x^0 + r_1(B_2)x^1 + r_2(B_2)x^2 = 1 + 4x + 2x^2$$

We then find the rook polynomial for the original board B by using Theorem 1.

$$R(x, B) = R(x, B_1)R(x, B_2) = (1+4x+3x^2)(1+4x+2x^2) = 1+8x+21x^2+20x^3+6x^4$$

In order to find the number of ways to assign the five different classes to the five professors using the rook polynomial $R(x, B)$, we apply the theorem from the Inclusion-Exclusion Principle. This theorem states that the number of ways to arrange n distinct objects when there are forbidden positions is

$$\sum_{k=0}^{n} (-1)^k r_k(B)(n-k)!$$

For our case, n=5. Thus,

$$\sum_{k=0}^{5}(-1)^k r_k(B)(5-k)! = 5! - 8(4!) + 21(3!) - 20(2!) + 6(1!) - 0(0!) = 20$$

Therefore, there are 20 ways in which we can assign the five classes to the five professors. □

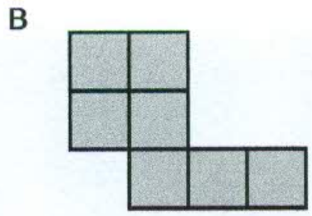
3 Calculus of Non-Disjoint Boards

Up until now we have seen boards that can be decomposed into disjoint subboards. What happens when a board cannot be decomposed?

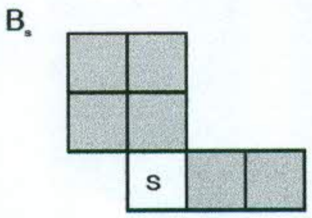
Theorem 3: Let B be any board of darkened squares. Let S be one of the squares of B, and let B_s and B_s^*. Then

$$R(x, B) = R(x, B_s) + xR(B_s^*)$$

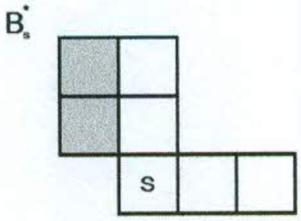
Example 2: Suppose we are given a non-disjoint board B:



We proceed by deleting a square on B, denoted by the letter S. We get the following board B_s.



Next, we delete all squares in the same row and column as S, to get the board B_s^*.



Applying theorem 1 to B_s we find the rook polynomial to be

$$R(x, B_s) = R(x, B_s')R(x, B_s'') = (1 + 4x + 2x^2)(1 + 2x)$$

The rook polynomial for B_s^* is

$$R(x, B_s^*) = 1 + 2x$$

Then applying theorem 3, we find the rook polynomial for B,

$$R(x, B) = R(x, B_s) + xR(x, B_s^*) = 1 + 7x + 12x^2 + 4x^3\square$$

Now that we have seen an example of the process, let us explain what is happening. If square S is not used, we must place k noncapturing rooks on B_s. If square S is used, then we must place $k-1$ noncapturing rooks on B_s^*[Tucker]. We then get a recurrence relation $r_k(B) = r_k(B_s) + r_{k-1}(B_s^*)$. Then,

$$R(x, B) = \sum_k^n r_k(B)x^k$$

$$= \sum_k^n r_k(B_s)x^k + \sum_k^n r_{k-1}(B_s^*)x^k$$

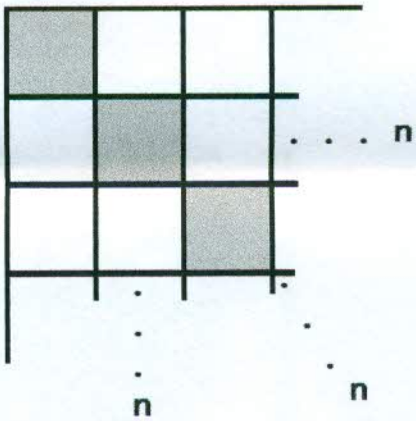
$$= \sum_k^n r_k(B_s)x^k + x\sum_h^n r_h(B_s^*)x^h = R(x, B) = R(x, B_s) + xR(B_s^*)$$

4 Derangements

Definition: A derangement is a permutation of elements in a set such that none of the elements appear in their original position. The number of derangements of a set of n elements is denoted by D_n.

Example 3: Suppose we have n balls labeled $1, 2, 3, ...n$ and these balls are in their respective positions $1, 2, 3, ...n$. What is the derangement of the balls? What is the probability of the derangement?

We begin by constructing an nxn board B, with forbidden positions along the diagonal.



We see that B has n disjoint subboards. The rook polynomial for all B_n's is $R(x, B_n) = 1 + x$. Thus,

$$R(x, B) = R(x, B_1)R(x, B_2)\cdots R(x, B_n)$$
$$= (1 + x)^n$$

By theorem 2, the number of ways to arrange n distinct objects is

$$\sum_{k=0}^{n}(-1)^k r_k(B)(n-k)!$$