TWELVE-FOLD WAY

We count functions $f : K \to N$

	arbitrary functions	**injective functions**	**surjective functions**
K dist N dist	n^k	$n^{\underline{k}} = P(n,k) = n(n-1)\cdots(n-k+1)$	$n!S(k,n)$
K indist N dist	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\left(\!\binom{n}{k-n}\!\right)$
K dist N indist	$\sum_{n=1}^{k} S(k,n) = B(k)$	1 **if** $k \le n$ 0 **if** $k > n$	$S(k,n)$
K indist N indist	$\sum_{n=1}^{k} p_n(k)$	1 **if** $k \le n$ 0 **if** $k > n$	$p_n(k)$

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(1) $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$ is “x lower factorial n” and consists of n factors starting with x and going down.

(2) $(x)^n = (x+(n-1)\cdots(x+2)(x+1)x$ is “x upper factorial n” consists of n factors starting with x and going up. This concept is used in connection with partitioned permutations (not part of the twelve-fold way). If k distinct objects are to be placed into n distinct boxes where the position in each box is also distinct, then there would be n places for the first object. Now regard that first object as sub-partitioning the box into which it went. The next object sees $n+1$ places, and so forth, so that the total number of placements is $(n)^k$.

(3) **Stirling numbers** of the first and second kinds are the coefficients that connect the powers of x with the lower factorial polynomials in x.

(i) If you write $(x)_k = \sum_{n=1}^{k} T(k,n)x^n$, the $T(k,n)$ are Stirling numbers of the first kind.

(ii) If you write $x^k = \sum_{n=1}^{k} S(k,n)(x)_n$, the $S(k,n)$ are Stirling numbers of the second kind.

Examples: $x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x$, so $T(4,1) = 1$, $T(4,2) = -6$, etc.

$x^4 = (x)_4 + 6(x)_3 + 7(x)_2 + (x)_1$, so $S(4,1) = 1$, $S(4,2) = 6$, etc.

Stirling numbers of the first kind can be negative, but Stirling numbers of the second kind are always positive. Stirling numbers of the first kind are related to the number of partitions of permutation cycles. Stirling numbers of the second kind directly enumerate the number of partitions of an k-set into n parts. The total number of partitions of a k-set is $B(k) = \sum_{n=1}^{k} S(k,n)$. This is a Bell number.

(4) A **partition of a set** is a family of non-void subsets that are mutually disjoint and exhaustive. Specifically, if S is a set, and $B_i(\neq \varnothing) \subset S$ for i in some index set Λ, then if $B_i \cap B_j = \varnothing$ whenever $i \neq j$, and $\bigcup_{i\in\Lambda} B_i = S$, we say $\{B_i : i \in \Lambda\}$ is a partition of S.

(5) A **composition** of a positive integer is an ordered sum. A k-composition is an ordered sum consisting of k summands. For example, a 3-composition of 7 is $2+1+5$. The dot and space argument shows that the number of k-compositions of n is $\binom{n-1}{k-1}$. This concept is related strongly to the counting of multisets.

(6) An k-**multiset on a set** S with n elements is a set created by choosing k elements from those n available in S with repeated selections permitted (unlike the formation of regular subsets). Multisets enter the picture when selections allowing replacement are considered. Technically, a multiset is itself a function from the elements of S into $\mathbb{N}$. The image of an element is the number of times it is selected. Think of a multiset like this: $\{a,a,a,b,b,c\}$ is a multiset on the set $\{a,b,c\}$. If S has n elements, then the number of multisets of size k we can make from them is the same as the number of solutions in nonnegative integers of $x_1 + x_2 + \cdots x_n = k$. The value of x_i is the number of times element i (out of n) is selected. Add 1 to each x_i and call it y_i, then $y_1 + y_2 + \cdots y_n = k+n$. Now the number of solutions of this equation in strictly positive integers (we got rid of zeroes) is the number of multisets of size k. But this is precisely the number of n-compositions of $k+n$. We know from the previous item that this number is $\binom{k+n-1}{n-1}$. We denote the number of k-multisets of an n-set

by $\left(\!\binom{n}{k}\!\right)$. Note that $\binom{k+n-1}{n-1} = \binom{k+n-1}{k} = \frac{n^{(k)}}{k!}$. The last equality says that the number of k-multisets of an n-set can be gotten by considering the number of partitioned permutations (arrangements) of a k-set into n parts and then allowing the elements of the k-set to be shuffled, necessitating a division by the number of such shuffles, $k!$.

(7) A **partition of an integer** is a special thing that differs from the partition of a set. Think of it as an unordered sum (as opposed to a composition of an integer which is an ordered sum). We have a standard way of talking about integer partitions....put the largest pieces first and forget any zeroes. So a partition of 8 into 4 parts would be $(3,3,1,1)$. Another notation is $\langle 3^2, 1^2 \rangle$, and we write $\vdash$ to indicate partitioning, as in $(3,3,1,1) \vdash 8$. In general the number of partitions of k into n parts is $p_n(k)$. These numbers appear in the last row of the twelve-fold way. There is no simple formula for them, and there is an enormous theory devoted to them.

(8) If we raise the expression $x_1 + x_2 + \cdots x_k$ to the n^{th} power, we need a generalization of the binomial formula: $(x_1 + x_2 + \cdots x_k)^n = \sum_{r=0}^{n} \sum_{s=0}^{n} \cdots \sum_{w=0}^{n} \binom{n}{r\ s\ \cdots\ w} x_1^r x_2^s \cdots x_k^w$ where $r + s + \cdots w = n$. For example

$(x+y+z)^3 = x^3 + 3x^2y + 3x^2z + 3xy^2 + 6xyz + 3xz^2 + y^3 + 3y^2z + 3yz^2 + z^3$

The term $\binom{n}{r\ s\ \cdots\ w} = \frac{n!}{r!s!\cdots w!}$ is called the multinomial coefficient, and it corresponds to the number of ways n objects can be arranged if r of them are of one type, s are of another type, and so forth. If there are n boxes and only $m = r + s + \cdots w$ balls, then the number of distinct arrangements is $\binom{n}{r\ s\ \cdots\ w} \frac{1}{(n-m)!}$.