Hypermath/Calculus

TABULAR INTEGRATION BY PARTS

BACKGROUND:

Problems requiring integration by parts often require one or several iterations of the method. Tabular integration gives a fast and simple way of organizing the results of successive integrations by parts.

KEY FACTS:

Recall that integration by parts is the "inverse" of the product rule for differentiation. Specifically, given functions $f(x)$ and $g(x)$ with respective antiderivatives $F(x)$ and $G(x)$, we note that by the product rule, $(F(x)G(x))' = f(x)G(x) + F(x)g(x)$. Integrating both sides and rearranging gives $\int F(x)g(x)dx = F(x)G(x) - \int f(x)G(x)dx$. The idea is to identify a part of the integrand that is itself easily integrable...$g(x)$...and hope that the complementary part...$f(x)$...can be differentiated down to a simpler function. So if $\int F(x)g(x)dx$ is difficult, but $\int g(x)dx$ and $\int f(x)G(x)dx$ are doable, the problem has been partially defeated. Very often $g(x)$ is $\sin x$, $\cos x$, or e^x.

TYPICAL PROBLEMS:

1) **The function which is successively differentiated becomes zero**.

Given $\int x^3 \sin x dx$, we identify x^3 as the function to be successively differentiated and $\sin x$ to be integrated. We construct the following table:

$\frac{d}{dx}$	$\int$
x^3	$\sin x$
$3x^2$	$-\cos x$
$6x$	$-\sin x$
6	$\cos x$
0	$*$

The result from the table is read by multiplying the first entry in the first column by the second entry in the second column, then continuing the pattern one cell below and so forth until zero is reached. The successive products are added with alternating signs beginning with +. So the example given would be $\int x^3 \sin x dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x\cos x = 3x^2 \sin x + (6x - x^3)\cos x$.

2) **The functions which are successively differentiated/integrated cycle back to their original form** (**to within a constant**).

Given $\int \sin ax \cos bx dx$, with $a \neq b$, we identify $\sin ax$ as the function to be successively differentiated and $\cos bx$ to be integrated. We construct the following table:

$\frac{d}{dx}$	$\int$
$\sin ax$	$\cos bx$
$a\cos ax$	$\frac{\sin bx}{b}$
$-a^2 \sin ax$	$\frac{-\cos bx}{b^2}$

The result from this table is read exactly like the preceding example except that, since zero is not going to be reached, the last term is the integral of the product of the two terms in the final row. So the last term in the second row always appears twice...once as a factor in a regular product and once as a factor in an integrand. The sign attached to the integral is the same one that would arise in the alternating scheme. Successive partial integrations where the complementary pieces of the integrand cycle back to their original form except for constants lead to the situation where the original and the last tabular integral can be merged to yield a solution. The given example would be $\int \sin ax \cos bx dx = \sin ax\left(\frac{\sin bx}{b}\right) - a\cos ax\left(\frac{-\cos bx}{b^2}\right) + \int(-a^2 \sin ax)\left(\frac{-\cos bx}{b^2}\right)dx$. This can be rearranged to give $\left(1 - \frac{a^2}{b^2}\right)\int \sin ax \cos bx dx = \frac{1}{b}\sin ax \sin bx + \frac{a}{b^2}\cos ax \cos bx$ from which it follows that $\int \sin ax \cos bx dx = \frac{1}{b^2 - a^2}(b \sin ax \sin bx + a\cos ax \cos bx)$.