Solution of Equations

Quadratic Case

A familiar staple of high school algebra, the quadratic formula, is easily derived by "completing the square". Given $ax^2 + bx + c = 0$, where $a \neq 0$ (or it wouldn't be a quadratic any longer), we can divide by a and arrange the terms as $x^2 + \frac{b}{a}x = -\frac{c}{a}$, then after adding $\left(\frac{b}{2a}\right)^2$ to each side, finally write $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$. The whole point of the addition is to create a perfect square on the left hand side. We have $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$, and upon taking the square root, we get $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$, from which the formula is immediate: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The ninth century Persian mathematician al-Khwārizmī had geometric methods to solve general monic quadratic equations.

A Simplification

The quadratic formula is simple enough that we retain the luxury of an arbitrary leading coefficient, but the formulas for higher degree equations are complicated enough that we prefer to work with monic polynomials, so we immediately convert the leading coefficient to 1 by division if necessary. The German mathematician Ehrenfried von Tschirnhausen discovered a further possible simplification in 1683. Von Tschirnhausen was a contemporary of Leibnitz and a man of many talents. He is best known as the discoverer of porcelain, at least the European variety. He was also one of the few mathematicians to offer a correct solution to the brachistochrone problem, also known as the problem of determining the curve of quickest descent.

Starting with a general monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, the von Tschirnhausen transformation $y = x + \frac{a_{n-1}}{n}$ eliminates the second-highest degree monomial. To see this expand $\left(y - \frac{a_{n-1}}{n}\right)^n + a_{n-1}\left(y - \frac{a_{n-1}}{n}\right)^{n-1} + \cdots + a_0$. The first few term are $y^n - n \cdot \frac{a_{n-1}}{n}y^{n-1} + a_{n-1}y^{n-1} + \big($terms of degree $n - 2$ or lower$\big)$. The transformed equation, sometimes called the "depressed equation", then looks like $y^n + b_{n-2}y^{n-2} + \cdots + b_0 = 0$. One can easily imagine that this discovery stimulated attempts to find clever substitutions that would eliminate other powers. However, starting with a depressed equation, the natural substitution that eliminates the y^{n-2} term unfortunately restores a y^{n-1} term, so the best we can do is play algebraic Whac-a-Mole. In the derivations below, we will start with depressed equations.

Cubic Case

A general solution to the cubic equation eluded the ancients and was only discovered by the Italian Renaissance mathematicians in the 1540's. The physicist Richard Feynman remarked that this was the first truly modern significant mathematical result. The method hinges on the counterintuitive introduction of an extra degree of freedom. Starting with $x^3 + px + q = 0$, we set $x = u + v$ to obtain $(u^3 + 3u^2v + 3uv^2 + v^3) + p(u + v) + q = 0$. Rearranging, we have $u^3 + v^3 + (u + v)(3uv + p) + q = 0$. Now we take away the extra degree of freedom and invoke the constraint $3uv + p = 0$ in order to get rid of the inconvenient term $(u + v)(3uv + p)$. This implies $v = \frac{-p}{3u}$, and the equation can now be written $u^3 + \left(\frac{-p}{3u}\right)^3 + q = 0$. Multiplying through by u^3, we have a quadratic in u^3, namely $(u^3)^2 + qu^3 - \frac{p^3}{27} = 0$, from which we deduce

$$u^3 = \frac{-q \pm \sqrt{q^2 - 4\left(-\frac{p^3}{27}\right)}}{2} = \frac{-q \pm \sqrt{\frac{4q^2}{4} + 4\left(\frac{p^3}{27}\right)}}{2} = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

The constraint $3uv + p = 0$ is symmetric in u and v, so we take $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ and

$v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$. Then

$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ in terms of the original coefficients. Now if $x = \alpha$ is one solution of the cubic equation, $\alpha\omega$ and $\alpha\omega^2$ will also be solutions, where $\omega = \frac{-1 + i\sqrt{3}}{2}$ (so $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$) is a complex cube root of 1, since $(\alpha\omega)^3 = (\alpha\omega^2)^3 = \alpha^3$.

As an example, suppose that $x^3 + 24x - 56 = 0$. Note that $x = 2$ is a root. Our formula gives $u = \sqrt[3]{-\frac{-56}{2} + \sqrt{\left(\frac{-56}{2}\right)^2 + \left(\frac{24}{3}\right)^3}} = \sqrt[3]{28 + \sqrt{1296}} = 4$ and

$v = \sqrt[3]{-\frac{-56}{2} - \sqrt{\left(\frac{-56}{2}\right)^2 + \left(\frac{24}{3}\right)^3}} = \sqrt[3]{28 - \sqrt{1296}} = -2$, so we recover $x = 4 - 2 = 2$. The other (complex) solutions to the equation are $-1 \pm i\sqrt{3}$. Note that our basic formula could easily return a complex number if $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0$. In that case, one of $\alpha\omega$ or $\alpha\omega^2$ would have to be real, since a cubic equation always has at least one real root.

The discovery and publication of the solution to the general cubic was a famous early case in intellectual property rights. Girolamo Cardano was a noted intellectual during the Italian Renaissance and author of numerous works on mathematics, astrology, medicine, and gambling. He seems to have been a contentious individual, and this kept him from securing any permanent position at a university, but he did have students. He was chronically broke, and supported himself largely by gambling, which led to his publication of a basic theory of probability. His contemporary and countryman Niccoló Fontana discovered the formula for the cubic about 1540 and disclosed it after some badgering to Cardano, who promised to keep it a secret. Several years later, Cardano came into possession of an unpublished manuscript by Scipione del Ferro, which predated by at least a decade the disclosure by Fontana. This manuscript indicated that del Ferro had independently discovered the cubic formula prior to Fontana. *(It was not uncommon for mathematicians in those days to keep their best discoveries secret, since they were always subject to "problem challenges" from their rivals. This was the mathematical equivalent of a duel. Each participant submitted to the other a problem they could solve but would be difficult for the recipient to solve. Being defeated in the challenge had ramifications for one's university position or patron support, so del Ferro had a good defense with his cubic formula.)* Cardano then felt that his promise to Tartaglia of non-disclosure was moot, and he published *Ars Magna* in 1545 containing the formula, giving both Fontana and del Ferro ample credit in the text for their independent discoveries. Nevertheless, this touched off a public dispute with Fontana that lasted a decade. Today, the formula we have derived above is called the Cardano-Tartaglia formula, giving both men credit. Tartaglia is not a new character here. Fontana had the misfortune of being an inhabitant of Brescia during a seige by the French, who decided to massacre the inhabitants as payback for rather effective resistance. A French soldier cut Fontana's jaw and palate badly enough that he found it difficult to speak, and thereby gained the nickname "The Stammerer", which in Italian is "Tartaglia". It is not clear why del Ferro does not receive more modern credit for this discovery.

We should mention that these Renaissance results were not as conceptually seamless as the modern version. Cardano was the first mathematician to work with negative numbers in a serious way, and although he was aware of imaginary numbers, he did not profess to understand their properties. The approach at the time was to separate cases based on the coefficients, and to work only towards real solutions (either one or three). For example, Cardano would have assumed that $p, q > 0$ and treated the cases $x^3 = px + q$, $x^3 + px = q$, and $x^3 = px + q = 0$ separately. His contribution was extending and systematizing the basic method of del Ferro and Fontana.

Quartic Case

The solution to the quartic case followed quickly on the heels of the Cardano-Tartaglia-del Ferro discovery of the cubic formula. The quartic formula was discovered in the late 1540's by Lodovico Ferrari, a student of Cardano. The depressed quartic lacks, of course, the cubic term, and this suggests trying to make a perfect square out of the fourth degree and second degree terms. Given $x^4 + ax^2 + bx + c = 0$, we can immediately write $x^4 + 2ax^2 + a^2 = (x^2 + a)^2 = -bx - c + ax^2 + a^2$. If the right hand side were a perfect square, we could take square roots and be well on our way to a solution. But we have no basis to assume that it is. Drawing from our experience with the cubic, we introduce a degree of freedom and try to force the right hand side to be a square. Consider the term $(x^2 + z + a)^2 = (x^2 + a)^2 + 2z(x^2 + a) + z^2$. Here we have introduced z with the intention of giving it a value later that will allow us to take square roots. Evidently we have $(x^2 + z + a)^2 = 2z(x^2 + a) + z^2 - bx - c + ax^2 + a^2$. The right hand side, as a standard quadratic in x, is $(2z + a)x^2 - (b)x + (2az + z^2 - c + a^2)$. This will be a perfect square of the form $(\alpha x + \beta)^2$ provided its discriminant $b^2 - 4 \cdot (2z + a) \cdot (2az + z^2 - c + a^2) = 0$. But this is a cubic equation in z, namely $8z^3 + 20az^2 + z(16a^2 - 8c) + (4a^3 - 4ac - b^2) = 0$. Let z_0 be a root of this so-called "resolvent cubic", then we can write $(x^2 + z_0 + a)^2 = (\alpha x + \beta)^2$, where $\alpha = \sqrt{2z_0 + a}$ and $\beta = \pm\sqrt{2az_0 + z_0^2 - c + a^2}$. The sign of β here must be taken opposite to that of b to ensure $-b = 2\alpha\beta$. Upon taking square roots, we have $x^2 + z_0 + a = \pm\left(x\sqrt{2z_0 + a} - sgn(b) \cdot \sqrt{2az + z_0^2 - c + a^2}\right)$. Finally we seek solutions of $x^2 \pm x\sqrt{2z_0 + a} + \left(z_0 + a \pm (-sgn(b))\sqrt{2az_0 + z_0^2 - c + a^2}\right) = 0$. In general these two quadratics will spin off two solutions each and there will be four solutions to the original quartic.

As an example, suppose that $x^4 - 10x^2 + 4x + 8 = 0$. We identify $a = -10$, $b = 4$, and $c = 8$. Then the resolvent cubic is $8z^3 - 200z^2 + 1536z - 3696 = 0$, which can be simplified to $z^3 - 250z^2 + 192z - 462 = 0$. The solutions to this cubic are $z_0 = 7, 9 \pm \sqrt{15}$. Then using $z_0 = 7$ and noting $sgn(b) = +$, we have $x^2 \pm x\sqrt{2 \cdot 7 - 10} + 7 - 10 \mp \sqrt{2 \cdot (-10) \cdot 7 + 7^2 - 8 + 1^2} = 0$. More recognizably, these quadratics are $x^2 + 2x - 4 = 0$ and $x^2 - 2x - 2 = 0$, so the solutions to the original quartic are then $1 \pm \sqrt{3}$ and $-1 \pm \sqrt{5}$. For practical calculations, it may be easier once z_0 is determined to immediately substitute everything into $(2z_0 + a)x^2 - (b)x + (2az_0 + z_0^2 - c + a^2)$ and obtain the perfect square that equals $(x^2 + z + a)^2$.

Here is a problem from Ars Magna Chapter XXXIX (Problem VI): Find a number which is equal to its square root plus twice its cube root. Let the number be x^6, then its square root is x^3 and its cube root is x^2, hence $x^6 = x^3 + 2x^2$, and we can reduce this to $x^4 = x + 2$, or $x^4 - x - 2 = 0$. Give that one a try.