from scratch series...........

THIRD ISOMORPHISM THEOREM

Let G **be a group where** $H \leq G$ and $N \triangleleft G$. **Then** $\frac{H}{H \cap N} \cong \frac{HN}{N}$.

BACKGROUND:

See the comments for the First Isomorphism Theorem.

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x, y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \triangleleft G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \triangleleft G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$. This is the *factor group* of G by H.

2) The kernel of a homomorphism is always a normal subgroup of the domain.

3) The intersection of two subgroups of a given group is also a subgroup.

4) If $H, N \leq G$ and $N \triangleleft G$, then $HN = \{hn : h \in H, n \in N\} \leq G$.

5) The natural map $\phi : G \to G/N$ given by $g \mapsto gN$ is a homomorphism.

6) First Homomorphism Theorem: If $\phi : G \to H$ is a group homomorphism. Then $G/\ker\phi \cong \operatorname{Im}\phi$.

PROOF STRATEGY:

The proof is a straightforward application of the First Isomorphism Theorem. Once we get the pieces defined properly, it is immediate. We will show that $\frac{HN}{N}$ is a factor group and invent a homomorphism from H to it that has $H \cap N$ as its kernel.

PROOF:

Draw a baseball diamond. Put $H \cap N$ at home, N at first, HN at second, and H at third. While you're at it, put G in center field. This is the classical "Diamond Isomorphism" proof diagram. We have the following inclusion relationships: $G \geq HN \geq N \geq H \cap N$ and $G \geq HN \geq H \geq H \cap N$. The theorem states that second base mod first base is isomorphic to third base mod home. OK..I'll lose the baseball metaphor now.

First we will attack the issue of legitimizing $\frac{HN}{N}$. We need to show $N \triangleleft HN$, or equivalently, $(hn)n_1(hn)^{-1} \in N$ for arbitrary $h \in H$ and $n, n_1 \in N$. Now $hn_1h^{-1} \in N$ because $N \triangleleft G$, hence $gn_1g^{-1} \in N$ for all $g \in G$...and every h is a g. This is a funny situation. Think about it...there is no containment relationship between H and N assumed. It could be that $H \cap N = G$ or $H \cap N = e_G$, or anything in between. We are going to use the normality condition without N being a corresponding normal subgroup of H. OK...back to $(hn)n_1(hn)^{-1} = h[nn_1n^{-1}]h^{-1} = hn_2h^{-1} \in N$. So indeed $N \triangleleft HN$ and the factor group $\frac{HN}{N}$ makes sense.

Perhaps you've noticed that the homomorphisms that work in the proofs of theorems of this type are pretty simple. Generally speaking, you would expect this, since the assumptions of the theorems are modest and there is no obvious reason to complicate things. But, of course, we speak with the benefit of hindsight. Anyway, here's another. Start with the natural homomorphism $\phi : G \to G/N$ given by $g \mapsto gN$. Now restrict it to H, so we have $\phi|_H : H \to G/N$. We're lazy (but well-meaning), so we try not to do any extra work in our proofs, and by invoking the natural map (known to be a homomorphism), we don't have to show that $\phi|_H$ is a homomorphism, since the restriction inherits that property.

Now what is the kernel of $\phi|_H$? Well, $\ker \phi|_H = \{h \in H : hN = N\}$, since N is the identity for G/N. By the elementary rules for cosets, whenever $hN = N$, we know $h \in N$, hence $\ker \phi|_H = H \cap N$.

OK...what is the image of $\phi|_H$? It is pretty clear that the restriction cuts out all gN where $g \notin H$, and leaves only cosets of the form hN for $h \in H$. But $\{hN : h \in H\} = \{hnN : h \in H, n \in N\} = \frac{HN}{N}$.

Finally, all of the pieces are available to plug into the *First* Isomorphism Theorem: $H / \ker \phi|_H \cong \operatorname{Im} \phi|_H$ in this context yields $\frac{H}{H \cap N} \cong \frac{HN}{N}$, as we intended.