from scratch series...........

SECOND ISOMORPHISM THEOREM

Let G **be a group where** $H \lhd G$, $N \lhd G$, and $N \subset H$. **Then** $\frac{G/N}{H/N} \cong G/H$.

BACKGROUND:

See the comments for the First Isomorphism Theorem. It looks superficially like we are "cancelling" the subgroup N by some arithmetic operation, whatever that would mean in this context (i.e. nothing!), although it is helpful as a mnemonic. In reality, we are forming three factor groups and observing that they stand in this relationship to one another by virtue of the First Isomorphism Theorem.

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x, y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \lhd G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \lhd G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$. This is the *factor group* of G by H.

2) The kernel of a homomorphism is always a normal subgroup of the domain.

3) First Homomorphism Theorem: If $\phi : G \to H$ is a group homomorphism. Then $G/\ker\phi \cong \operatorname{Im}\phi$

PROOF STRATEGY:

The proof is a straightforward application of the First Isomorphism Theorem. Once we get the pieces defined properly, it is immediate. We will invent a homomorphism from G/N to G/H that has H/N as its kernel.

PROOF:

First, notice that $N \subset H$ and $H \triangleleft G$ imply $N \triangleleft H$. Why? We are given $N \triangleleft G$, which means $gng^{-1} \in N$. But every $h \in H$ is also a g, so $hnh^{-1} \in N$ as well. So it makes sense to form the factor group H/N. Now the factor groups G/H and G/N are available immediately from Hölder's Theorem, and the only thing left to do is explain why the three of them may be configured as we claim. One loose end at the moment is the expression $\frac{G/N}{H/N}$. In order for it to make sense, we have to establish that $H/N \triangleleft G/N$. Let's try to hit two birds with one stone. If we can dream up a homomorphism from G/N that has H/N as its kernel and G/H as its image, not only will the "factor group of factor groups" $\frac{G/N}{H/N}$ be well-defined, but the *First* Homomorphism Theorem will immediately give us the fact that it is isomorphic to its image G/H, which is the entire claim of the Second Isomorphism Theorem.

OK... here is a candidate mapping...$\phi : G/N \to G/H$ which sends the coset gN to the coset gH. The mapping just strips out the coset representative g, tosses away N, then applies g to the subgroup H to form a bigger coset gH. We better establish that this mapping is actually a homomorphism before we go any further. To do so, we need to verify that $\phi(g_1Ng_2N) = \phi(g_1N)\phi(g_2N)$. Well, on the left hand side normality allows us to write $\phi(g_1Ng_2N) = \phi(g_1[g_2N]N) = \phi(g_1g_2N) = g_1g_2H$. And on the right hand side $\phi(g_1N)\phi(g_2N) = (g_1H)(g_2H) = g_1[Hg_2]H = g_1[g_2H]h = g_1g_2H$. So ϕ is a homomorphism and we are in business.

What is the kernel of ϕ? It is whatever gets mapped to the identity of G/H, which is, of course, just H. Consider what ϕ does to the coset $hN \in G/N$. It sends it to $hH = H$! So all the elements (and only those) of H/N, namely the cosets of the form hN, are mapped onto H. This allows us to conclude immediately that $H/N \triangleleft G/N$ by the second key fact above. Now $\frac{G/N}{H/N}$ is a legitimate construction with $\ker\phi = H/N$.

What is the image of ϕ? Certainly any coset gH has a pullback gN, so $\operatorname{Im}\phi = G/H$. The First Isomorphism Theorem gives us that $G/N / \ker\phi \cong \operatorname{Im}\phi$, or $\frac{G/N}{H/N} \cong G/H$.