from scratch series...........

FIRST ISOMORPHISM THEOREM

Let $\phi : G \to H$ **be a group homomorphism**. **Then** $G \diagup \ker\phi \cong \mathrm{Im}\,\phi$

BACKGROUND:

Camille Jordan was apparently aware of this theorem in a limited context by 1870, but the first time anyone wrote it down in the form we recognize today was in a 1927 paper by Emmy Noether. Hölder's Theorem, which says that if a subgroup is normal its cosets have a natural operation under which they form a group themselves...the factor group...was not available until 1888. The First Isomorphism Theorem is sometimes considered to be part of the Fundamental Homomorphism Theorem for groups, which subsumes the first, second, and third isomorphism theorems. That is convenient, since there does not seem to be any agreement after eighty years on which of the second two parts is to be called the Second Isomorphism Theorem, and which is the Third Isomorphism Theorem. You will see them labelled either way, depending on the textbook. As with many "first, second, third" theorems, the first one is the most significant and the others depend on it. The Fourth Isomorphism Theorem, also known as Zassenhaus' Lemma or the Butterfly Theorem from the Hasse diagram of its constituent groups, is not part of this bundle.

The First Isomorphism Theorem creates an isomorphism where there wasn't one by restricting the codomain to the range of a given homomorphism, thus ensuring surjectivity, and "modding out" the kernel of that homomorphism to obtain injectivity. Fortunately, the image of the domain group under a homomorphism is a subgroup of the codomain group, and the kernel of a homomorphism is a normal subgroup of the domain group, so the factor group construction makes sense. The punchline of the theorem is that the image group and factor group are essentially the same.

KEY DEFINITIONS:

1) A *group homomorphism* is an operation preserving mapping between two groups, specifically if G and H are groups and $\phi : G \to H$, then ϕ is a homomorphism iff for all $x, y \in G$ it is true that $\phi(xy) = \phi(x)\phi(y)$. Another way to state the operation preserving property is to say that the group operations and the mapping commute.

2) The *kernel* of a group homomorphism is the preimage or pullback of the identity in the codomain group. Arbitrary maps between groups don't necessarily have kernels...only homomorphisms, so that the operation preserving feature of the homomorphism ensures that the kernel is a subgroup. If $\phi : G \to H$ is a homomorphism, then $\ker\phi = \{g \in G : \phi(g) = e_H\}$, where e_H is the identity in H.

3) A group *isomorphism* is a bijective group homomorphism. Group isomorphisms give all of the algebraic properties of the group on one side of the mapping to the group on the other side. Be careful to realize that properties like order and boundedness are not algebraic and do not need to be preserved by isomorphisms.

4) A *coset* is formed by taking an arbitrary element of a group, and operating with that element on every element in a chosen subgroup. If the arbitrary element operates from the left, we get a left coset of the subgroup, and if it operates from the right we get a right coset. It turns out that cosets of a given subgroup are mutually disjoint and all the same size.

5) A subgroup H is *normal* in the group G (written $H \triangleleft G$) if for any $g \in G$ and $h \in H$, the product $ghg^{-1} \in H$. Intuitively speaking, *normality* measures how mixed up the elements of a subgroup can become when they are operated first on the left by an arbitrary element of the group

and then on the right by its inverse. If you get back to the subgroup every time, then the subgroup is normal (an older word is "invariant", which explicitly conveys this stability property). There is no guarantee that you will, so this is a special situation and it has many useful consequences. Another intuitive viewpoint is that the elements of H "almost commute" with the elements of G, that is $gh = h'g$.

KEY FACTS:

1) Hölder's Theorem : If $H \triangleleft G$, then $G/H = \{gH : g \in G\}$ is a group where the operation is given by $(xH)(yH) = xyH$

2) The kernel of a homomorphism is a normal subgroup of the domain

PROOF STRATEGY:

This proof is constructive...we build an isomorphism out of the raw material available. The most important thing we have going for us is the operation preserving map...the homomorphism...which might be turned into an isomorphism if the defects in its bijectivity can be repaired. Repairing surjectivity is easy, just restrict the codomain to the image. Repairing injectivity is less obvious, but we collapse all elements in the domain that get mapped to a common element in the range by the expedient of forming cosets of the kernel. Cosets of the kernel can't distinguish such elements, and as far as the cosets are concerned the mapping is injective.

PROOF:

Since $\ker\phi \triangleleft G$, it makes sense to consider the factor group $G/\ker\phi$. Since we are looking for a natural association between cosets of this kernel and elements of H, I propose that we investigate the mapping $\Psi : G/\ker\phi \to H$, given by $x\ker\phi \to \phi(x)$. This is surely the most obvious guess. In words, this mapping sends a coset to the image of its representative. Now there might be a problem here if we picked two different representatives for the same coset and they wind up being mapped to distinct range elements. Then we wouldn't even have a function. But our fears are unfounded, since if $x\ker\phi = y\ker\phi$, then $\phi(x) = \phi(x)\phi(\ker\phi) = \phi(x\ker\phi) = \phi(y\ker\phi) = \phi(y)\phi(\ker\phi) = \phi(y)$. This chain of equalities is justified by the operation preserving property of the mapping ϕ and the fact that $\phi(\ker\phi) = e_H$. We can now assert that our new mapping Ψ is at least well-defined....i.e. is a function!

Is Ψ injective? If $\phi(x) = \phi(y)$, then $\phi(x)[\phi(y)]^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1}) = e_H$, but this means $xy^{-1} \in \ker\phi$. In turn it must be that $x \in y\ker\phi$, hence $x\ker\phi \subset y\ker\phi$. Reversing the roles of x and y, we have $y\ker\phi \subset x\ker\phi$, so $y\ker\phi = x\ker\phi$, and we have established that the mapping Ψ is injective.

Now any injective map is bijective onto its range, and clearly the range of Ψ is precisely the range of ϕ. So trimming the codomain from H down to $\operatorname{Im}\phi$ makes Ψ bijective.

Finally, is Ψ operation preserving? In other words is $\Psi(xHyH) = \Psi(xH)\Psi(yH)$. Well, $\Psi(xHyH) = \Psi(xyH) = \phi(xy) = \phi(x)\phi(y) = \Psi(xH)\Psi(yH)$, so yes.

Now we have established that Ψ is a well-defined operation preserving bijection, which is sufficient to declare Ψ to be an isomorphism, so $G/\ker\phi \overset{\Psi}{\cong} \operatorname{Im}\phi$.

Recall that the homomorphic image of a group is always a group as well, but it may be far from isomorphic to the original...in fact it could collapse a really big group into the identity. This theorem allows us to manufacture an isomorphism where there was none.