DISCRETE DISTRIBUTIONS

1) Name: Bernoulli distribution

Situation: Single trial with binary outcome (success/failure), experiments like this are called Bernoulli trials.

Notation: x is number of successes, θ is probability of success.

Formula: $f(x;\theta) = \theta^x(1-\theta)^{1-x}$

Use: not too interesting practically, but more as a theoretical tool...it is the binomial distribution when $n = 1$.

2) Name: Binomial distribution

Situation: Repeated independent Bernoulli trials

Notation: x is number of successes, n is number of attempts, or trials, θ is probability of success on one trial (same for all and no trial influences any other).

Formula: $b(x;n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ for exactly x successes

$B(x;n,\theta) = \sum_{i=0}^{x}\binom{n}{i}\theta^i(1-\theta)^{n-i}$ for x or fewer successes. This is the cumulative binomial distribution (the cdf corresponding to the pmf).

Mean: $n\theta$

Variance: $n\theta(1-\theta)$

Use: finding the probability that after n identical and independent binary trials there will have been exactly x successes (for the pmf) or at least x successes (for the cdf).

3) Name: Binomial waiting time distribution or negative binomial distribution (often now called the Pascal distribution)

Situation: Repeated independent Bernoulli trials

Notation: x is number of trials required so that the k^{th} success occurs exactly on the x^{th} trial, θ is probability of success on one trial (same for all and no trial influences any other).

Formula: $b^*(x;k,\theta) = \binom{x-1}{k-1}\theta^k(1-\theta)^{x-k}$. Clearly the number of trials has to be at least the number of desired success, so $x \geq k$. This is an infinite distribution since the k^{th} success could be delayed

indefinitely if θ is small. The corresponding cdf is not commonly used, but represents the probability that the k^{th} success occurs sometime on or before the x^{th} trial.

Mean: $\frac{k}{\theta}$

Variance: $\frac{k}{\theta}\left(\frac{1}{\theta} - 1\right)$

Use: finding the probability that the k^{th} success occurs on the x^{th} trial.

4) Name: Geometric distribution

Situation: Repeated independent Bernoulli trials

Notation: x is number of trials required so that the first success happens on the x^{th} trial, θ is probability of success on one trial (same for all and no trial influences any other).

Formula: $g(x;\theta) = \theta(1-\theta)^{x-1}$. This is the Pascal distribution with $k = 1$.

Mean: $\frac{1}{\theta}$

Variance: $\frac{1}{\theta}\left(\frac{1}{\theta} - 1\right)$

Use: finding the probability that the first success occurs on the x^{th} trial.

5) Name: Hypergeometric distribution

Situation: Repeated Bernoulli trials that are not independent because the outcomes of previous trials affect the likelihood of subsequent outcomes.

Notation: x is number of successes, n is the number of trials required to achieve x successes, N is the pool of available trials of which M result in successes.

Formula: $h(x;n,N,M) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$. Clearly $x \le M$, $x \le n$, and $n - x \le N - M$ for this to work.

Mean: $\frac{nM}{N}$

Variance: $\frac{nM(N-M)(N-n)}{N^2(N-1)}$

Use: finding the probability of x successes in n trials where each trial is a pick from the pool of N available trials of which M are regarded as successes. As successes and failures are chosen from the pool, they are not replaced, so subsequent probabilities are affected. This distribution comes up in "sampling without replacement". Sampling with replacement is modeled by the binomial distribution.

6) Name: Poisson distribution

Situation 1: The Poisson distribution is an approximation to the binomial distribution for the case when n is large and θ is small. Before calculators, this was very important.

Situation 2: If $n \to \infty$ while $\theta \to 0$ in such a way that $n\theta = \lambda$, a constant, the Poisson distribution becomes exact in its own right and models the likelihood that x successes will occur in some unit of time or space given that the average number of successes per unit of time or space is λ.

Notation: x is the number of successes, λ is the average number of successes per unit of time or space

Formula: $p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$ for exactly x successes

$P(x;\lambda) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$ This is the cumulative Poisson distribution and gives the probability for at least x successes.

Mean: λ

Variance: λ (this is not a typo)

Use: Estimating the probability that a certain number of successes will happen given an average rate of them happening

7) Name: Uniform distribution

Situation: Each of n values of the random variable X are equiprobable

Notation: X is a random variable with finite range of cardinality n

Formula: $u(x) = \frac{1}{n}$

Mean: $\frac{1}{n} \sum_{k=1}^{n} x_k = \mu$

Variance: $\frac{1}{n} \sum_{k=1}^{n} (x_k - \mu)^2$

Use: Mostly pedagogical as this is unlikely unless we are modeling some simple phenomenon to illustrate a theoretical point (like throwing a single die)