ACTUARIAL NOTATION

1) $v(s,t)$ discount function - this is a function that takes an amount payable at time t and re-expresses it in terms of its implied value at time s. Why would its implied value be different? Because of the time preference for money. A dollar today is usually considered to be worth more than a dollar next year because something can be done with the dollar today to yield more than a dollar in a year. Time s can be before t, in which case the function really does discount (reduce in value), or it can be after t, in which case the discount function is often referred to as an accumulation or growth function. When s is today ($s = 0$), the convention is to write $v(0,t)$ as $v(t)$, and if $v(t)$ is a constant, just v. Key formulas are: (i) $v(s,t)v(t,u) = v(s,u)$, (ii) $v(s,t) = [v(t,s)]^{-1}$.

2) d_k discount rate - this is not the discount function, but it is closely related. $v(k,k+1)$ is the value at time k of 1 unit accruing at time $k+1$. This is less than one, so there is a decrement or loss in value when the future amount is restated in terms of present value. The decrement amount is the discount rate, hence $d_k = 1 - v(k,k+1)$. Note it is expressed as a positive quantity.

3) i_k interest rate - this is a quantity that in a sense is inverse to the disccount rate. The value at time $k+1$ of 1 unit accruing at time k is $v(k+1,k)$. This is greater than one, so there is an increment or gain in value. The increment amount is the interest rate, hence $i_k = v(k+1,k) - 1$. Recalling that $v(k+1,k) = [v(k,k+1)]^{-1}$, any one of the three quantities d_k, i_k, and $v(k,k+1)$ can be expressed in terms of any other. Key formulas are: (i) $d_k = \dfrac{i_k}{1+i_k}$, (ii) $i_k = \dfrac{d_k}{1-d_k}$.

4) $\mathbf{c} = \langle c_0, c_1, \cdots, c_k, \cdots, c_n \rangle$ cashflow vector - this is a new notation favored by the author to summarize a discrete cash stream, where c_0 is an amount accruing at time 0, and in general c_k accrues at time k. Positive entries correspond to money received and negative entries to money paid. Obviously the same cash stream event can be viewed from either the payor or payee's standpoint, and the cashflow vectors for each would have equal but opposite components. The vector notation works well with summations.

5) $Val_j(\mathbf{c};v)$ value of the cashflow vector $\mathbf{c}$ at time j relative to the discount function v - this is what the cash stream represented by the cashflow vector is worth at various times. $Val_0(\mathbf{c};v)$ is the present value of the cash stream. Key formulas are: (i) $Val_j(\mathbf{c};v) = \sum_{k=1}^{n} v(j,k)c_k$ - here each piece of the income stream is being adjusted to its

value at time j and they are all added up. Some elements of the cash stream grow (when $j > k$) and some are diminished (when $j < k$). (ii) the value function is linear.. $Val_j(\alpha\mathbf{c} + \beta\mathbf{d}; v) = \alpha Val_j(\mathbf{c}; v) + \beta Val_j(\mathbf{d}; v)$

6) $\ddot{a}(\mathbf{c}; v)$ this is the same as $Val_0(\mathbf{c}; v)$. Traditionally, actuaries denoted an annuity by a and an annuity due (payment made immediately at the beginning of the first time period) by $\ddot{a}$. If the first payment of an annuity due is zero, then it is really just a regular annuity. The author uses this gimmick to retain the traditional notation by considering all annuities to be due, but when they are actually regular annuities, the understanding is that $c_0 = 0$. If the discount function is constant, it is often suppressed, so you may see $\ddot{a}(\mathbf{c})$.

7) $\ddot{a}(1_n)$ the 1_n means that 1 appears as every component in the cashflow vector. So $1_n = \langle 1, \cdots, 1\rangle$. Since v does not appear explicitly, we take it as a constant. Key formula: $\ddot{a}(1_n) = \frac{1 - v^n}{1 - v}$, which comes from the geometric progression sum formula.

8) $\ddot{a}(\mathbf{j}^n)$ $\mathbf{j}^n$ is the vector $\langle 1, 2, \cdots, n-1, n\rangle$, which represents a cash stream that is an arithmetic progression. In pre-spreadsheet days, cash streams that had regular patterns to them were important because there were simple formulas to evaluate their present values, such as the formula in (7) and the one immediately below. Spreadsheets have made regularity less of an issue these days as the calculations can be done easily by brute force (cut-and-paste programming). Key formula: $\ddot{a}(\mathbf{j}^n) = \frac{\ddot{a}(1_n) - nv^n}{1 - v}$.

9) ${}_k\mathbf{c}$ and ${}^k\mathbf{c}$ this notation is intended to allow splitting of a cash stream into the part before time k and the part after time k. The part at precisely time k goes with the "after". This convention is standard but a little confusing...settlements occuring at time k (payments in or out) are not figured into the valuation of a balance at time k. This is the way they do it in the insurance business...banks have a slightly different approach (see (12)). If $\mathbf{c} = \langle c_0, c_1, \cdots, c_k, \cdots, c_n\rangle$, then ${}_k\mathbf{c} = \langle c_0, c_1, \cdots, c_{k-1}, 0, \cdots, 0\rangle$ and ${}^k\mathbf{c} = \langle 0, 0, \cdots 0, c_k, \cdots, c_n\rangle$. You might remember this better if you associate "subscript down" with beginning and "subscript up" with end. Key formula: $\mathbf{c} = \mathbf{c}_k +^k \mathbf{c}$, which is nothing more than the definition.

10) $B_k(\mathbf{c}; v)$ this is the balance of the cash stream given by $\mathbf{c}$ at time k. It is the value of all payments made up to (but not including, see(9)) time k, restated with their imputed values at that time. Specifically, $B_k(\mathbf{c}; v) = \sum_{j=1}^{k-1} v(k,j)c_j$, which may be recognized as $Val_k({}_k\mathbf{c}; v)$. Again, the discounting pegs all values to time k, but the actual payment at time k is excluded.

11) ${}_kV(\mathbf{c};v)$ this is the reserve implied by the cash stream at time k. It is the value of all payments that are still to be made discounted back to time k. The normal situation is that these payments represent an obligation on the part of whomever owns this particular cashflow vector (recall there are two viewpoints for every cashflow vector). Since the reserve consists of cash owed, we take the negative and write ${}_kV(\mathbf{c};v) = -\sum_{j=k}^{n} v(k,j)c_j$, which is the same as $Val_k({}^k\mathbf{c};v)$. Key formulas: (i) $Val_k(\mathbf{c};v) = Val_k({}_k\mathbf{c};v) - Val_k({}^k\mathbf{c};v)$, or $Val_k(\mathbf{c};v) = B_k(\mathbf{c};v) -_k V(\mathbf{c};v)$ (ii) if $\mathbf{c}$ is a cashflow vector that is zero at all times (for example, when we try to determine internal rates of return), then $B_k(\mathbf{c};v) -_k V(\mathbf{c};v) = 0$ for all k, and we can see that the balance and reserve are always equal.

12) $\widetilde{B}_k(\mathbf{c};v)$ this is the augmented balance, more commonly used for loans. We just append the payment at time k to the balance at time k. So $\widetilde{B}_k(\mathbf{c};v) = B_k(\mathbf{c};v) + c_k$.

13) the $\circ$ operator it may be useful to shift the time origin for a cashflow calculation, and it that case we write $\mathbf{c} \circ k$ to indicate that we have discarded the first $k-1$ components of $\mathbf{c} = \langle c_0, c_1, \cdots, c_k, \cdots, c_n \rangle$ and we are now working with $\langle c_k, \cdots, c_n \rangle$. Note carefully that this is not the same as ${}^k\mathbf{c} = \langle 0, 0, \cdots, 0, c_k, \cdots, c_n \rangle$...the dimension of the vector has been reduced by k. So componentwise, $(\mathbf{c} \circ k)_j = c_{k+j}$. The time-shift operator may also be applied to discount functions, where the shifted discount function $v(n+k, m+k) = v \circ k(m,n)$. This is the author's notation...it could be simplified to $v'(m,n) = v(n+k, m+k) = v \circ k$, which makes more intuitive sense to me.

14) $a_{\overline{n}|}$ this notation is traditional and hard to read....the subscript is $\overline{n}|$, which is a right angle symbol enveloping the n. The angle symbol means duration, so $a_{\overline{n}|}$ is a unit cash stream that extends for n time periods and has no payment initially. We have $a_{\overline{n}|} = \ddot{a}(0, 1_n)$. In the event that there is an initial payment, we put the two dots over the angle-subscripted a as well - $\ddot{a}_{\overline{n}|} = \ddot{a}(1_n)$.

15) $s_{\overline{n}|}$ actuaries use s to represent accumulated (summed) value. $s_{\overline{n}|} = Val_n(0, 1_n)$ and similarly with the two dots.

16) ℓ_x this is the number of people alive on their x^{th} birthday. Our author adds the additional qualification that these people were alive on their zeroth birthday (live births), but that seems redundant to me. One assumption is that we have a fixed initial cohort.

17) d_x this is the number of people who were alive on their x^{th} birthday but died before their $(x+1)^{st}$ birthday. So $d_x = \ell_x - \ell_{x+1}$. Note that ℓ_x regarded as a function of x

is monotone decreasing.

18) ${}_n p_x$ this is the probability that a person aged x will live to be $n+x$. Key formula: ${}_n p_x = \frac{\ell_{x+n}}{\ell_x}$..this is the ratio of the death-depleted cohort to its original size. Whenever $n = 1$, it is customary to write p_x and suppress the 1. Note that by dividing by the original cohort size, the absolute population counts are reduced to probabilities, and our calculations are uncoupled from the census particulars. The ℓ_x numbers are facts on the ground that pertain to specific populations. It is a research and statistical problem to assemble an accurate picture of the pattern of mortality in a population. Once this is done, however, the implied probabilities can be presented in a mortality table as a starting point for our life actuary calculations.

19) ${}_n q_x$ this is the probability that a person aged x will not live to be $n+x$. They may die anytime during the intervening interval of n years. Certainly ${}_n p_x +_n q_x = 1$. Whenever $n = 1$, it is customary to write q_x and suppress the 1.

20 ω this is the lowest age at which everyone is assumed to have died. There may still be some survivors with age $\omega - 1$. Usually it is taken as 120 years.

21) (x) for talking purposes, a generic person of age x is denoted by (x)

22) e_x this is life expectancy at age x, defined to be the average number of years that (x) may expect to live beyond his present age. So his expected age at death will be $x + e_x$. Technically, this is the *curtate* life expectancy to differentiate it from the *complete* life expectancy, denoted by $\mathring{e}_x$, which accounts for the fact that not everyone is cooperative enough to die on their exact birthday. A reasonable assumption is that deaths among individuals of age x before they become $x+1$ are uniformly distributed. Hence the mean of that distribution is $\frac{1}{2}$ year, and we add that to the curtate to obtain the complete life expectancy: $\mathring{e}_x = e_x + \frac{1}{2}$. Calculation of e_x is intuitively clear...starting with a cohort of size ℓ_x, they will as a group live ℓ_{x+1} years the first year, then this group will account for ℓ_{x+2} more years lived, and so forth. It all stops when we get to $\ell_{\omega-1}$, since anyone still alive will die before age ω. Hence the total number of additional years lived by the original cohort is $\sum_{k=1}^{\omega-1} l_{x+k}$. Dividing by the size of the original cohort gives the average number of additional years lived by anyone in the cohort: $e_x = \frac{1}{\ell_x} \sum_{k=1}^{\omega-1} l_{x+k} = \sum_{k=1}^{\omega-1} \frac{\ell_{x+k}}{\ell_x}$. But $\frac{\ell_{x+k}}{\ell_x} = {}_k p_x$, so $e_x = \sum_{k=1}^{\omega-1} {}_k p_x$. Note that younger people have higher life expectancies, but older people have higher expected ages at death. The popular understanding of life expectancy is that it is e_0, and this is used as a quality of life measure in comparing countries.

23) $\ddot{a}_x(\mathbf{c})$ this is the single premium today for an annuity on the life of (x) with payout vector $\mathbf{c}$. $\ddot{a}_x(\mathbf{c}) = \sum_{k=0}^{\omega-x-1} c_k v(k)_k p_x$. You can understand this formula as follows. (x) may live another $\omega - x - 1$ years at most, so the payments to him at each of those years must appear in the sum. (x) is scheduled to receive c_k in year k. The present value of such a payment is $c_k v(k)$. Moreover, there is a probability that (x) will not live to his $(x+k)^{th}$ birthday, in which case the insurer will not have to make that payment. The ${}_k p_x$ constitute a discrete probability distribution that (x) will survive during year k...we know they sum to 1 because (x) is not immortal. The expected value of the payment stream is then precisely the various amounts $(c_k v(k))$ weighted by their respective probabilities $({}_k p_x)$, summed over all possible years. Individually, $c_k v(k)_k p_x$ is the current premium that would have to be charged on average to the recipient (x) for a pure endowment of c_k, payable, of course at year k.

24) $y_x(k,n)$ interest and survivorship discount function. It combines the influence of the time value of money with the mortality of the recipient. It is defined as $\frac{v(n)_n p_x}{v(k)_k p_x}$. Conceptually, it is the ratio of the mortality-corrected discount function at year n to the mortality-corrected discount function at k. So when we discount an amount using $y_x(k,n)$ where $k < n$, say, we get a little reduction in the uncorrected rate due to the ratio of mortalities, which itself is going to be less than 1.