CALCULUS 3 - FALL 2017 - HOMEWORK 1 -Solutions

1) Differentiate $(\sin x)^{\cos x}$

$(\sin x)^{\cos x} = e^{(\ln \sin x)\cos x}$*, so (using* $(e^{g(x)})' = e^{g(x)} \cdot g'(x)$*) we have* $(e^{(\ln \sin x)\cos x})' = e^{(\ln \sin x)\cos x} \cdot [(\ln \sin x)\cos x]' = (\sin x)^{\cos x} \cdot \left[\dfrac{\cos^2 x}{\sin x} - \sin x(\ln \sin x) \right]$

2) Differentiate $(2x^3 + x^2 - 6x + 10)^5(5x^4 - 3x^2 + 1)^8(\tan^2 x - \sin^3 x)^{11}$

Using $f'(x) = f(x)[\ln(f(x))]'$ *we get* $\left[(2x^3 + x^2 - 6x + 10)^5(5x^4 - 3x^2 + 1)^8(\tan^2 x - \sin^3 x)^{11} \right] \cdot$ $[5\ln(2x^3 + x^2 - 6x + 10) + 8\ln(5x^4 - 3x^2 + 1) + 11\ln(\tan^2 x - \sin^3 x)]'$*, which in turn is*

$$\left[(2x^3 + x^2 - 6x + 10)^5(5x^4 - 3x^2 + 1)^8(\tan^2 x - \sin^3 x)^{11} \right] \cdot$$
$$\left[\frac{5(6x^2 + 2x - 6)}{2x^3 + x^2 - 6x + 10} + \frac{8(20x^3 - 6x)}{5x^4 - 3x^2 + 1} + \frac{11(2\tan x \sec^2 x - 3\sin^2 x \cos x)}{\tan^2 x - \sin^3 x} \right]$$

3) Find $\int \tan\theta d\theta$

$$\int \tan\theta d\theta = \int \frac{\sin\theta d\theta}{\cos\theta} = -\int \frac{d(\cos\theta)}{\cos\theta} = -\ln(\cos\theta) = \ln(\sec\theta) + C$$

4) Find the smallest positive x where $\sin x = \cos x$ and find the acute angle between $y = \sin x$ and $y = \cos x$ at that point

The desired point is $x = \dfrac{\pi(8n+1)}{4}$ *with* $n = 0$*, or* $\dfrac{\pi}{4}$*. The slope of* $y = \sin x$ *at* $\dfrac{\pi}{4}$ *is* $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$*. The slope of* $y = \cos x$ *at* $\dfrac{\pi}{4}$ *is* $-\sin\dfrac{\pi}{4} = -\dfrac{\sqrt{2}}{2}$*. Slopes are tangent (functions), so the angle the sine curve makes with the horizontal at* $\dfrac{\pi}{4}$ *is* $\arctan\left(\dfrac{\sqrt{2}}{2}\right) = 62$ *degrees and for the cosine* $\arctan\left(-\dfrac{\sqrt{2}}{2}\right) = -62$ *degrees, so the total angle is* 124 *degrees. So the smaller of the two pairs of vertical angles formed by the intersecting tangents (lines) is* $\dfrac{360 - 2 \cdot 124}{2} = 56$ *degrees.*

5) Show how to find the area of an ellipse with semi-axes a and b.

Consider the part of the ellipse in the first quadrant. Since $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$*,* $y = b\sqrt{1 - \dfrac{x^2}{a^2}}$.

Then by symmetry the area A *of the entire ellipse is* $4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$*. Now let* $\frac{x}{a} = \cos\theta$*, then* $dx = -a\sin\theta d\theta$*. Then the integral becomes* $4b\int_{\frac{\pi}{2}}^0 \sin\theta(-a\sin\theta d\theta) = 4ab\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$*. Now let* $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ *(half angle formula). Note that* $\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}d\theta = \int_0^{\frac{\pi}{2}} \frac{d\theta}{2} - \int_0^{\frac{\pi}{2}} \frac{\cos 2\theta}{2}d\theta = \frac{\pi}{4} + 0$*. So the overall result is* $4ab \cdot \frac{\pi}{4} = \pi ab$*.*

6) Find $\lim_{x\to 0+} x^x$

Note that $x^x = e^{x\ln x}$*, and since the exponential is continuous on the real line,* $\lim_{x\to 0+} x^x = e^{\lim_{x\to 0+} x\ln x}$*. As* x *goes to zero thru positive numbers,* $x\ln x$ *tends to* $0 \cdot (-\infty)$*. So write* $x\ln x = \frac{\ln x}{\frac{1}{x}}$ *which approaches* $-\frac{\infty}{\infty}$*. Then L'Hôpital gives the limit as* $\frac{1/x}{-1/x^2} = -x$*, so* $\lim_{x\to 0+} x\ln x = 0$ *and therefore* $\lim_{x\to 0+} x^x = 1$*.*