ABSTRACT ALGEBRA 2 - SPRING 2017 - ASSIGNMENT 2 (Sylow Theory) - Solutions

1) Show that conjugacy is an equivalence relation

(i) $\forall x \in G,\ exe^{-1} = e$ *for the identity* e*, so any element is self-conjugate (reflexivity)*
(ii) If $gxg^{-1} = y$*, then by multiplying we get* $g^{-1}yg = x$*, so if* x *is conjugate to* y*, then* y *is conjugate to* x *and conversely, so conjugation is symmetric*
(iii) If $gxg^{-1} = y$ *and* $hyh^{-1} = z$*, then* $h(gxg^{-1})h^{-1} = (hg)x(hg)^{-1}$ *so conjugation is transitive*
These three properties make a relation an equivalence relation

2) If G is a group and $|G| = 2n - 1$ for $n > 1$, show that $x^{-1} \notin cl(x)$

Suppose for the sake of contradiction that $x^{-1} \in cl(x)$*, then* $gxg^{-1} = x^{-1}$ *for some* $g \in G$*. So for an arbitrary conjugate* hxh^{-1}*, taking its inverse we have* $(hxh^{-1})^{-1} = h^{-1}x^{-1}h = h^{-1}(gxg^{-1})h = (h^{-1}g)x(h^{-1}g)^{-1}$*, which belongs to* $cl(x)$*. So* hxh^{-1} *and* $(hxh^{-1})^{-1}$ *both belong. They must be distinct, otherwise* hxh^{-1} *would have order* 2*, which can't happen because the element order would have to divide* $2n - 1$*. But if all conjugates and their inverses are present as distinct pairs, it must be that* $|cl(x)|$ *is even. Since* $|cl(x)| = [G : C_G(x)]$ *and* $|G| = [G : C_G(x)]|C_G(x)|$*, it follows that the even* $|cl(x)|$ *must divide the odd* $|G|$*. The contradiction shows* $x^{-1} \notin cl(x)$*.*

3) Find all 3-SSGs in G if $|G| = 48$

We did this in class.

4) Show that every group of order 56 has a proper normal subgroup

$56 = 2^3 \cdot 7$*.* $N(2) = 2n + 1$ *(ie odd) and* $2n + 1$ *divides* 7*. So* $N(2) = 1$ *or* 7*. If* $N(2) = 7$*, There are* $7 \cdot 7$ *non-identity elements, then that leaves room for a single* $7 - SSG$ *which must be normal since it is unique. If If* $N(2) = 1$*, we are done immediately, since the* $2 - SSG$ *would be normal.*

5) What is the smallest composite integer (not prime and greater than 1) such that there is a unique group of that order?

Admissible composites in order are $2 \cdot 3,\ 2 \cdot 4,\ 3 \cdot 3,\ 2 \cdot 5,\ 3 \cdot 4,\ 2 \cdot 7,\ 3 \cdot 5,\ \cdots$. *We can eliminate* 6 *(*$\mathbb{Z}_3$ *and* S_3*),* 8 *(*$\mathbb{Z}_8$, D_4, *etc),* 9 *(*$\mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*),* 10 *(*$\mathbb{Z}_{10}$, D_5*),* 14 *(*$\mathbb{Z}_{14}$, D_7*),* 15 *(*3 *does not divide* $5 - 1$, *so by Thm 24.6 there is only one group of order* $3 \cdot 5$*...bingo)*

6) Show that if $|G| = 175$, then G is abelian

$175 = 5^2 \cdot 7$, *so consider* $N(5) = 5n + 1$. *This divides* 7 *only if* $n = 0$, *so* $N(5) = 1$, *and we see the* $5 - SSG$ *is unique, hence normal. Call it* H. *Similarly,* $N(7) = 7m + 1$, *and this divides* 5 *only if* $m = 1$, *hence the* $7 - SSG$ *is unique, hence normal. Call it* K. *Note* $H \cap K = \{e\}$, *and* $G = HK$, *so* $G = H \times K \cong H \oplus K$. *Since both* H *and* K *are abelian (*H *has order* p^2 *and* K *is cyclic) we know* $H \oplus K = G$ *is abelian.*