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Problem 1850

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Let $\mathcal{T}$ be a topology on a finite set X. Define a topology on X to be regular if for any nonempty closed $E \subseteq X$ and $x \in X \setminus E$, there exist disjoint open sets U and V in $\mathcal{T}$ such that $E \subseteq V$ and $x \in U$. Prove or disprove that the topological space $(X, \mathcal{T})$ is regular if and only if $\mathcal{T}$ has a base $\mathcal{B}$ which is a partition of X.

Solution:

The statement is true.

(Sufficiency) Suppose that $\mathcal{T}$ has a base $\mathcal{B}$ which is a partition of X. Given $x \notin F \subset X$, with F closed, we observe that $x \in F^c$, which is open. By assumption, there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subset F^c$. Now $X \setminus B_x$ is exactly the union of all mutually disjoint partition elements in $\mathcal{B}$ except B_x, and hence is open since $\mathcal{B}$ is a base. But then $F \subseteq X \setminus B_x$ and we have separation of x and F by the disjoint open sets B_x and $X \setminus B_x$, respectively, which is regularity.

(Necessity) On the other hand, suppose that $(X, \mathcal{T})$ is regular. For each $x \in X$, define $U_x = \bigcap\{O \in \mathcal{T} : x \in O\}$. The finiteness of $(X, \mathcal{T})$ guarantees that each U_x is open. Let $R \subset X \times X$ be the relation determined by $(x,y) \in R$ if $y \in U_x$. We claim this is an equivalence relation. Certainly $(x,x) \in R$. Suppose for the sake of contradiction that $(x,y) \in R$ but $(y,x) \notin R$, that is $y \in U_x$.but $x \notin U_y$. By regularity, there is a separation by disjoint open sets U and V of the point y and the closed set U_y^c ,which contains x. Say $y \in U$ and $x \in U_y^c \subset V$. Then V is represented among the open sets defining U_x by intersection, and it follows that $y \notin U_x$. The contradiction establishes that R is symmetric. Moreover, by definition of U_x and U_y as intersections, symmetry implies $U_x \subseteq U_y$ and $U_y \subseteq U_x$, hence $U_x = U_y$. Finally, suppose that (x,y) and (y,z) belong to R. We have $U_x = U_y = U_z$, so $(x,z) \in R$ and we have transitivity. Having established that R is an equivalence relation, a set of representatives $\mathcal{B}$ of the equivalence classes of X modulo R partitions X. Given $z \in X$ and any open set $O \in \mathcal{T}$ with $z \in O$, there is some $B \in \mathcal{B}$ with $z \in B$. By construction, $z \in B(= U_z) \subseteq O$, which shows $\mathcal{B}$ is a base for $\mathcal{T}$.

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