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Problem 1836

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Let $n \geq 3$ be a natural number. Find how many pairwise non-congruent triangles are there among the $\binom{n}{3}$ triangles formed by selecting three vertices of a regular n-gon.

Solution:

The number of non-congruent triangles formed in the given manner is $nint\left(\frac{n^2}{12}\right)$, where $nint(\bullet)$ is the nearest integer function. Note first that an equivalence class of triangles under congruence is completely determined by the unordered lengths of the three sides, and also the distance between any two vertices of a regular n-gon is determined injectively by the smaller of the two numbers of intervening sides. A triangle formed in the prescribed manner defines an unordered partition of n into three parts in a natural way. Start at an arbitrary vertex and count consistently in either direction the number of sides of the n-gon until the next triangle vertex is encountered, then the next, and finally back to the first. There is an obvious induced bijection between congruence classes and partitions of n into three parts. Surjectivity of the induced map is clear, and distinct congruence classes induce different partitions, so injectivity follows. The partition function $p_3(n)$ then enumerates the congruence classes.

Note: The partition function $p_3(n)$ has ordinary generating function $\frac{x^3}{(1-x)(1-x^2)(1-x^3)} = \sum_{n=3}^{\infty} p_3(n)x^n = x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + 12x^{12} + \cdots$, but $p_3(n)$ can be written most simply as $nint\left(\frac{n^2}{12}\right)$ [Honsberger, R., *Mathematical Gems III*, MAA, pp. 40-45 1985.]

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