Mathematics Magazine

Problem 1795

Proposed by Jeff Groah, Montgomery College, Conroe, TX]

Find a function $f : [0,1] \to [0,1]$ such that for each nontrivial interval $I \subseteq [0,1]$, we have $f(I) = [0,1]$.

Solution:

We will cast our argument in terms of binary expansions, and this requires us to first dispose of the matter of ambiguity of representation. For example, $1.\overline{0}$ and $0.\overline{1}$ both represent 1. To ensure uniqueness of representation, we will agree to use the expansion with fewer 1's. More specifically, if $x \in [0,1]$ has a binary representation $b_0.b_1 \cdots b_j \cdots$ with $b_i = 0$ and $b_j = 1$ for $j > i$, then we will use the equivalent representation with $b_i = 1$ and $b_j = 0$ for $j > i$.

For $x \in [0,1]$ define $f(x) = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \sum_{i=1}^{n} b_i$. The disambiguation step above makes f well-defined, and we show that f has the desired property. The idea of the construction is to first use an initial block of binary digits to locate a domain value in a given interval, and then to use the remaining tail of the binary expansion to generate a desired range value without disturbing the preceding condition.

Given the nontrivial interval $I = [a,b] \subseteq [0,1]$, we may find a suitable k such that $2^{-k} < b - a$. Then a partition of $[0,1]$ into subintervals of length 2^{-k} will necessarily produce a subinterval $[c,d] \subseteq [a,b]$ where both c and d are 2-adic numbers with binary expansions having all zeroes after the k^{th} place. Suppose then that $c = 0.c_1 \cdots c_k \overline{0}$.and consider $t = 0.0 \cdots 0 t_{k+1} t_{k+2} \cdots$, where the t_i are completely arbitrary for $i > k$. Let $\alpha = c + t$ and observe that $\alpha \in [c,d]$, since $t < 2^{-k}$. (equality is ruled out by the disambiguation rule).

Now $\alpha = 0.c_1 \cdots c_k t_{k+1} t_{k+2} \cdots$ and $f(\alpha) = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \left(\sum_{i=1}^{k} c_i + \sum_{i=k+1}^{n} t_i \right) = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \sum_{i=k+1}^{n} t_i$, since deleting a finite number of terms will not affect the limit superior. We may arbitrarily fix the proportion of t_i's that are 1's in a particular α in order to have $f(\alpha) = r \in \mathbb{Q} \cap [0,1]$. Writing r in lowest terms as $\frac{p}{q}$, where $p \leq q$, set $\alpha_r = 0.c_1 \cdots c_k \overline{11 \cdots 10 \cdots 0}$ with the repeating block of q digits containing exactly p 1's. Evidently $f(\alpha_r) = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \sum_{i=k+1}^{n} t_i = \overline{\lim_{m \in \mathbb{N}}} \frac{1}{mq} \sum_{i=k+1}^{mq} t_i = \overline{\lim_{m \in \mathbb{N}}} \frac{mp}{mq} = r$.

Finally, any $\lambda \in \mathbb{Q}^c \cap [0,1]$ may be expressed as the limit of a weakly increasing sequence of rationals $\{r_1, r_2 \cdots\}$. In this case, with $r_i = \frac{p_i}{q_i}$ set $\alpha_\lambda = 0.c_1 \cdots c_k d_{k+1} \cdots d_{k+q_1} e_{k+q_1+1} \cdots$ where the block of digits labelled d_i has q_1 entries of which p_1 are 1's, the block labelled e_i has q_2 entries of which p_2 are 1's, and so on. We claim $f(\alpha_\lambda) = \lambda$. Fix $\epsilon > 0$. Then there exists an N such that $n > N$ implies $\lambda - r_n < \epsilon$. Now $f(\alpha_\lambda) = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \sum_{i=1}^{n} b_i = \overline{\lim_{n \in \mathbb{N}}} \frac{1}{n} \sum_{i=i(\epsilon)}^{n} b_i$, where we have neglected not only the locating digits (up thru index k) but also the digits belonging to the finite number of blocks associated with the rationals with index less than or equal to N Again, the limit superior is not affected by removal of a finite number of terms. The reduced sum starts with the first digit, indexed by $i(\epsilon)$, of the block of length q_{N+1} corresponding to the smallest rational differing from λ by less than ϵ. In other words, we only keep the digits in the blocks corresponding to the rationals that are within ϵ of λ. The proportion of 1's in any such block is at least r_{N+1} and never greater than λ. It follows that the limit superior of this sum is exactly λ. We have shown that $[0,1] \supseteq f(I) \supseteq f([c,d]) = [0,1]$, which establishes that $f(I) = [0,1]$.

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Thus

$$x_1^{\frac{x_1+x_2+\cdots+x_n}{x_1}} \geq x_1 + x_2 + \cdots + x_n$$

$$x_2^{\frac{x_2+x_3+\cdots+x_n}{x_2}} \geq x_2 + x_3 + \cdots + x_n$$

$$\vdots$$

$$x_{n-1}^{\frac{x_{n-1}+x_n}{x_{n-1}}} \geq x_{n-1} + x_n$$

$$x_n \geq x_n.$$

Adding these inequalities gives the desired result.

Also solved by Robert Calcaterra, Minh Can, Chip Curtis, Knut Dale (Norway), Robert L. Doucette, Dmitry Fleischman, Marty Getz and Dixon Jones, Eugen J. Ionascu, Tom Jager, Hidefumi Katsuura, Evangelos Mouroukos (Greece), Paolo Perfetti, Gabriel T. Prăjitură, Phillip P. Ray, Toufic Saad, C. R. Selvara and Suguna Selvaraj, Albert Stadler (Switzerland), and the proposer.

A many-to-one function **April 2008**

1795. *Proposed by Jeff Groah, Montgomery College, Conroe, TX.*

Find a function $f : [0, 1] \to [0, 1]$ such that for each nontrivial interval $I \subseteq [0, 1]$, we have $f(I) = [0, 1]$.

I. *Solution by Vadim Ponomarenko, San Diego State University, San Diego, CA.*
Each $x \in [0, 1]$, can be expressed in base 3: $x = [x_0.x_1x_2x_3\ldots]_3$, where each $x_i \in \{0, 1, 2\}$ and the representation does not end in an infinite string of 2s. If the base three representation has no digits 1 or infinitely many digits 1 among $x_1, x_2, \ldots$ then define $f(x) = 1$. (so $f(1) = 1$.) If there are a positive finite number of digits 1 among $x_1, x_2, \ldots$, find d so that $x_d = 1$ and $x_k \neq 1$ for $k > d$, and define

$$f(x) = [0.x'_{d+1}x'_{d+2}x'_{d+3}\cdots]_2 \qquad \text{where} \qquad 0' = 0,\ 2' = 1,$$

and we consider the result as a number in base 2. Each element of $[0, 1)$ has a preimage in the interval $[a, b]$, for any $a = 0.x_1\ldots x_d$ and $b = a + 3^d$, with $x_d = 1$. In fact, each element of $[0, 1]$ has a preimage in any interval of this type. (Note that $f(a + 3^d/2) = f(0.x_1\ldots x_d 11\ldots) = 1$.) Now, let I be a nontrivial interval. Then there is a $k > 0$ so that $[c, c + 3^k] \subseteq I$, for some $c = [0.y_1\ldots y_k]_3$, where each $y_i \in \{0, 1, 2\}$. We set $a = c + 3^{-k-1}$ and $b = a + 3^{-k-1}$. Then $[a, b]$ is an interval of the desired type, with $I \subseteq [a, b] \subset [c, c + 3^{-k}] \subseteq I$. This completes the construction.

II. *Solution by Jerrold W. Grossman, Oakland University, Rochester, MI.*
Let $\aleph$ be the cardinality of $U = [0, 1]$. The set J of nontrivial subintervals of U has cardinality $\aleph \cdot \aleph = \aleph$. Viewing $\aleph$ as an ordinal number (and noting that this implies that every initial segment of $\aleph$ has strictly smaller cardinality), we have a well-ordering $y_1, y_2, \ldots, y_\omega, \ldots$ of U and a well-ordering $I_1, I_2, \ldots, I_\omega, \ldots$ of J. Now by double transfinite induction define f as follows. For each $i \in \aleph$, for each $j \in \aleph$, choose a number $x_j \in I_j$ that has not yet been assigned a value and set $f(x_j) = y_i$; it is possible to find such a number because at each stage in the process the set of numbers that have so far been assigned a value has cardinality smaller than $\aleph$ and so does not exhaust I_j. For any $x \in U$ that has not been assigned a value upon completion of this process, arbitrarily set $f(x) = x$. The desired property of f is clear.

Also solved by Michael Andreoli, Michel Bataille, Tom Beatty, Michael W. Botsko, Paul Budney, Bruce S. Burdick, Robert Calcaterra, Elliott Cohen, A. K. Desai and K. V. Thaker, Marty Getz and Dixon Jones, Michael Goldenberg and Mark Kaplan, Eugen J. Ionascu, Jean-Christophe Laugier (France), Evangelos Mouroukos, Stephen