Mathematics Magazine

Problem 1686

Proposed by Shahin Amrahov, Ari College, Turkey.
Find all positive integer solutions (x,y) *to the equation* $2y^2 = x^4 + 8x^3 + 8x^2 - 32x + 15$.

Solution:

The only positive integral solution is $(3,12)$. Note that x must be odd, so substituting $x = 2n-1$ we get $y^2 = 8n^4 + 16n^3 - 20n^2 - 28n + 24$, where $n \geq 1$. Clearly $2 \mid y$, so setting $y = 2m$, we get $m^2 = (n-1)(n+2)(2n^2+2n-3)$. One, but not both, of the first two terms on the right is even, and the third is odd.. This implies that $m = 2t$ for $t \geq 0$ and then either $4 \mid n-1$ or $4 \mid n+2$. Consider the two cases:

Suppose $4 \mid n-1$. Then setting $n = 4s+1$, we have $t^2 = s(4s+3)(32s^2+24s+1)$, where $s \geq 0$. Assume for the sake of contradiction that $s \neq 0$. Now 3 is not a quadratic residue mod 4, so the factorization of $4s+3$ must contain some prime, say p, to an odd power. This prime must divide either s or $32s^2+24s+1$. The latter subcase is impossible, since that would require p to divide $(32s^2+24s+1)-(8s)(4s+3) = 1$. Now since $p \mid s$, we have $p \mid 3(s+1)$, so either $p = 3$ or $p \mid (s+1)$. Apparently $p = 3$, since $(s, s+1) = 1$. Setting $s = 3k$ and $t = 3w$, we may divide by 9 to obtain $w^2 = k(4k+1)(288k^2+72k+1)$. Each of the three terms on the right must now be a perfect square, since only the first two terms can have a common factor, namely 3, and $4k+1$ must contain an even power of 3. So let $k = u^2$, then $4u^2+1 = v^2$, where $u, v \geq 0$. It follows that $v^2 - 4u^2 = 1$, which is only possible if $u = 0$ and $v = 1$. It follows that $s = 0$, which contradicts the assumption that $s \neq 0$. Hence $s = 0$, and it follows that $y = 0$ and $x = 1$. We note that $(1,0)$ is a solution, but not of the required type.

Suppose $4 \mid n+2$. Setting $n = 4s-2$, we obtain $t^2 = s(4s-3)(32s^2-24s+1)$, where $s \geq 1$. For the sake of contradiction, assume $s > 1$. Again considering the quadratic residue, $4s-3$ must contain some prime p to an odd power, and this prime cannot divide $32s^2-24s+1$, or else it would divide $(32s^2-24s+1)-(8s)(4s-3) = 1$. Hence $p \mid s$ and $p \mid 3(s-1)$. It is again clear that $p = 3$. Now setting $s = 3k$ and $t = 3w$, we may divide by 9 to obtain $w^2 = k(4k-1)(288k^2-72k+1)$, where the three terms on the right are necessarily perfect squares, as before. Let $k = u^2$, then $4u^2-1 = v^2$, and it follows that $4u^2 - v^2 = 1$, which has no integer solutions. The contradiction establishes that the only possibility is $s = 1$, which translates into $n = 2$, hence $x = 3$, and we verify that the corresponding $y = 12$.

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2–12–04

1715. *Proposed by Barthel Wayne Huff, Salt Lake City, UT.*

An urn contains 35 red balls, labeled $1, 2, \ldots, 35$, and k blue balls. Balls are drawn one at a time at random, identified by number, then replaced, until a blue ball is drawn. Some red balls may be drawn more than once before the first blue is drawn. What is the minimal value of k for which the expected number of repetitions is less than 1?

Quickies

Answers to the Quickies are on page 73.

Q947. *Proposed by Robert Gregorac, Iowa State University, Ames, IA.*

Let

$$P_n(x) = x^n + x^{n-1} - x^{n-2} - x^{n-3} + x^{n-4} + \cdots \pm x \pm 1$$

be a polynomial with the pattern of sign changes indicated. (The last two terms depend on n modulo 4.) Prove that for $n \geq 4$, $P_n(x)$ must have at least one nonreal zero.

Q948. *Proposed by Yaroslav Krylyuk, Santa Fe Community College, Gainesville, FL.*

In $\triangle ABC$, angles B and C are acute and the altitude from A meets $\overline{BC}$ in K. Let M be an arbitrary point on $\overline{AK}$, let Q be the intersection of line BM with $\overline{AC}$, and let P be the intersection of line CM with $\overline{AB}$. Prove that $\angle PKA \cong \angle QKA$.

Solutions

A Positive Solution **February 2004**

1686. *Proposed by Shahin Amrahov, Ari College, Turkey.*

Find all positive integer solutions (x, y) to the equation

$$2y^2 = x^4 + 8x^3 + 8x^2 - 32x + 15.$$

Solution by Richard K. Guy, The University of Calgary, Alberta, Canada.

Since $2y^2$ is even, x is odd. Hence, $2y^2$ is a multiple of 8 and y is even. Substitute $x = 2X + 1$ and $y = 2Y$ to obtain

$$Y^2 = X(X+3)\big[2X(X+3) + 1\big]$$

As $X(X+3)$ and $2X(X+3)+1$ have no common factor other than 1, each is a square. But if $X > 1$, $X(X+3)$ lies strictly between $(X+1)^2$ and $(X+2)^2$ and so cannot be a square. So $X = 0$ or $X = 1$. The former choice leads to $y = 0$ and the latter to $x = 3$, $y = 12$, the only positive solution.

This completes the solution to the problem. Note, however, that if we write $x = v/(v-6)$ and $y = 6w/(v-6)^2$, then the equation becomes

$$w^2 = v^3 + v^2 - 84v + 270.$$

This is curve 1344Q1 in Cremona's tables. Its torsion points, ∞ and $(5, 0)$, correspond to $(1, 0)$ and $(-5, 0)$ on the original curve. The rank is 1 and a generator is $(9, 18)$, which corresponds to the solution $(3, 12)$. There are infinitely many rational solutions but, surprisingly, they are confined to the infinite component—there are none on the loop shaped finite component. Correspondingly, the rational points on the original curve, whose axes of symmetry are $y = 0$ and $x = -2$, are dense on the infinite components and absent from the loop. Examples of such points on the original curve are $(-7, \pm 12)$ and $(269/119, \pm 101160/119^2)$.

Also solved by JPV Abad, Roy Barbara (Lebanon), Michel Bataille (France), Brian D. Beasley, Tom Beatty, J. C. Binz (Switzerland), Jean Bogaert (Belgium), Brain Bradie, Stan Byrd and Lucas Van der Merwe, Robert Calcaterra, Minh Can, Michael Caulfield, John Christopher, Con Amore Problem Group (Denmark), Charles K. Cook, Randall J. Covill, Knut Dale (Norway), Charles R. Diminnie, Daniele Donini (Italy), Robert L. Doucette, Ragnar Dybvik (Norway), Timothy Eckert, Habib Y. Far, Tim Flood, Kenneth Fogarty, John F. Goehl, Michael Goldenberg and Mark Kalpan, G.R.A.20 Problems Group (Italy), Brian Hogan, Tom Jager, Kenneth Korbin, Victor Y. Kutsenok, Elias Lampakis (Greece), Kee-Lai Lau (China), Peter W. Lindstrom, Robert S. Lubarsky, David E. Manes, Allen J. Mauney, Northwestern University Math Problem Solving Group, Rolf Richberg (Germany), Fary Sami, Heinz-Jürgen Seiffert (Germany), Raul A. Simon (Chile), Albert Stadler (Switzerland), Daniel Stock, Paul Weisenhorn (Germany), Chu Wenchang and Di Claudio Leontina Veliana (Italy), Doug Wilcock, Hongbiao Zeng, Li Zhou, and the proposer. There was one incorrect submission.

Please, You Go First **February 2004**

1687. *Proposed by Sung Soo Kim, Hanyang University, Ansan Kyunggi, Korea.*

A two-player game starts with two sticks, one of length n and one of length $n+1$, where n is a positive integer. Players alternate turns. A turn consists of breaking a stick into two sticks of positive integer lengths, or removing k sticks of length k for some positive integer k. The player who makes the last move wins. Which player can force a win?

Solution by Li Zhou, Polk Community College, Winter Haven, FL.

Let A be the set of game positions with an even number of sticks and at most one stick of even length. Let B be the set of game positions with an odd number of sticks and at most two sticks of even length. Note that the initial and final game positions are in A. It is also evident that any move from a position in A results in a position in B, and that from each position in B a move can be made that results in a position in A. Thus the second player can force a win by always making a move that puts the game back into a position in A.

Also solved by JPV Abad, Roy Barbara (Lebanon), Jean Bogaert (Belgium), Glenn Bookhout, Con Amore Problem Group (Denmark), Timothy Eckert, David Gove, G.R.A.20 Problems Group (Italy), Jerrold W. Grossman, Richard K. Guy (Canada), Eric Harclerode, Tom Jager, Victor Y. Kutsenok, Robert S. Lubarsky, Jacob McMillen, Mike Pinter, Robert P. Sealy, Harry Sedinger, Furman Smith, Jon Stadler, Patrick A. Staley, Paul Weisenhorn (Germany), Hongbiao Zeng, and the proposer. There was one incorrect submission.

A Leading Multiple of p **February 2004**

1688. *Proposed by Mihai Manea, Princeton University, Princeton, NJ.*

Let p be an odd prime, and let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{p-1}x^{p-1}$ be a polynomial of degree $p-1$ with integral coefficients. Suppose that $p \nmid (P(b) - P(a))$ whenever a and b are integers such that $p \nmid (b-a)$. Prove that $p \mid a_{p-1}$.

Solution by David Gove, California State University Bakersfield, Bakersfield, CA.

Because $a \not\equiv b \pmod{p}$ implies $P(a) \not\equiv P(b) \pmod{p}$, it follows that $P(0)$, $P(1)$, $\ldots$, $P(p-1)$ form a complete residue system modulo p. Thus

$$0 \equiv \sum_{i=0}^{p-1} i \equiv \sum_{j=0}^{p-1} P(j) \equiv pa_0 + \sum_{k=1}^{p-1} a_k \left(\sum_{j=0}^{p-1} j^k \right) \pmod{p}. \qquad (*)$$

To evaluate the inner sum consider two cases. If $k = p-1$, then by Fermat's Theorem,

$$\sum_{j=0}^{p-1} j^k \equiv \sum_{j=1}^{p-1} 1 \equiv p-1 \pmod{p}.$$