Mathematics Magazine

Problem 1662

Proposed by Erwin Just (Emeritus) and Norman Schaumberger (Emeritus), Bronx Community College, Bronx NY.

Let x_k, $1 \le k \le n$, be positive real numbers with $\sum_{k=1}^{n} x_k^{2k-1} \le n$. Prove that $\sum_{k=1}^{n} (2k-1)x_k \le n^2$.

Solution:

By induction on n. The case for $n = 1$ is certainly true, so assume for any set $\{x_k\}_{1 \le k \le m}$ it is true that $\sum_{k=1}^{m} x_k^{2k-1} \le m$ implies $\sum_{k=1}^{m} (2k-1)x_k \le m^2$. Now given $\sum_{k=1}^{m+1} x_k^{2k-1} \le m+1$, either $x_k = 1$ for all k or there exists an $x_{k_0} < 1$. In the first instance, $\sum_{k=1}^{m+1} (2k-1)(1) = m^2$.

In the second instance, we may write $x_{k_0}^{2k_0-1} + \sum_{k=1}^{m} x_k^{2k-1} \le m+1$, where the sum has been re-indexed to allow the removal of k_0. But this requires $\sum_{k=1}^{m} x_k^{2k-1} \le m$, and by the induction hypothesis we have $\sum_{k=1}^{m} (2k-1)x_k \le m^2$. Letting $k_0 = m+1$, the re-re-indexed sum is $\sum_{k=1}^{m+1} (2k-1)x_k \le m^2 + [2(m+1) - 1] = (m+1)^2$, validating the induction. The two cases establish the general result.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL
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1690. *Proposed by Costas Efthimiou, Department of Physics, and Peter Hilton, Department of Mathematics, University of Central Florida, Orlando, FL.*

Prove that there exist functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\big(x - f(y)\big) = f(x) + y$$

for all $x,\ y \in \mathbb{R}$, and show how such functions can be constructed.

Quickies

Answers to the Quickies are on page 75.

Q937. *Proposed by Bill Chen, Philadelphia, PA, Clark Kimberling, Evansville, IN, and Paul R. Pudaite, Glen Ellyn IL.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \left\lfloor \frac{n}{k} + \frac{1}{2} \right\rfloor - \sum_{k=1}^{n} \left\lfloor \frac{n}{k+\frac{1}{2}} \right\rfloor = n.$$

Q938. *Proposed by William P. Wardlaw, U. S. Naval Academy, Annapolis, MD.*

Let R be a ring, let G be a finite subset of R that forms a multiplicative group under the multiplication of R, and let s be the sum of the elements of G. Prove that if G has more than one element, then s is either zero or a zero divisor in R. Give examples in which s is a nonzero divisor of zero.

Solutions

A Square Bound **February 2003**

1662. *Proposed by Erwin Just (Emeritus) and Norman Schaumberger (Emeritus), Bronx Community College of the City of New York, Bronx, NY.*

Let x_k, $1 \le k \le n$, be positive real numbers with $\sum_{k=1}^{n} x_k^{2k-1} \le n$. Prove that $\sum_{k=1}^{n}(2k-1)x_k \le n^2$.

I. *Solution by Michael G. Neubauer, California State University, Northridge, CA.*

Bernoulli's Inequality states that if $r \ge 1$ and $x \ge 0$, then $x^r - 1 \ge r(x-1)$. Replace x by x_k and r by $2k-1$, then do some rearranging to obtain

$$(2k-1)x_k \le x_k^{2k-1} - 1 + (2k-1).$$

It follows that

$$\sum_{k=1}^{n}(2k-1)x_k \le \sum_{k=1}^{n} x_k^{2k-1} - n + \sum_{k=1}^{n}(2k-1) \le n - n + n^2 = n^2.$$

II. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

We prove the following generalization:

Let I be a real interval containing 1 and let $f_k : I \longrightarrow \mathbb{R}$, $1 \le k \le n$, be differentiable and convex on I. If c is a real number, and $x_k \in I$, $1 \le k \le n$, with

$\sum_{k=1}^{n} f_k(x_k) \le c$, then

$$\sum_{k=1}^{n} f_k'(1)x_k \le c + \sum_{k=1}^{n} \left(f_k'(1) - f_k(1)\right).$$

The result in the problem statement follows by taking $f_k(x) = x^{2k-1}$, $x_k \in I = (0, \infty)$, and $c = n$.

To establish the generalization, first observe that for $1 \le k \le n$, the function g_k defined by $g_k(x) = f_k'(1)x - f_k(x)$ satisfies

$$g_k'(x) \ge 0 \qquad x \in I \text{ and } x < 1$$
$$g_k'(x) \le 0 \qquad x \in I \text{ and } x \ge 1,$$

so $g_k(x) \le g_k(1)$ for all $x \in I$. Hence

$$\sum_{k=1}^{n} f_k'(1)x_k \le c + \sum_{k=1}^{n} \left(f_k'(1)x_k - f_k(x_k)\right)$$
$$= c + \sum_{k=1}^{n} g_k(x_k) \le c + \sum_{k=1}^{n} g_k(1) = c + \sum_{k=1}^{n} \left(f_k'(1) - f_k(1)\right).$$

Also solved by Reza Akhlaghi, Tsehaye Andebrhan, Michael Andreoli, Carl Axness (Spain), Michel Bataille (France), Jean Bogaert (Belgium), Cal Poly Pomona Problem Solving Group, Minh Can, Mario Catalani (Italy), Con Amore Problem Group (Denmark), Knut Dale (Norway), Daniele Donini (Italy), Robert L. Doucette, Peter Drianov (Canada), Aaron Dutle, FGCU Problem Group, Ovidiu Furdui, G.R.A.20 Problems Group (Italy), Julien Grivaux (France), Enkel Hysnelaj (Australia), The Ithaca College Solvers, Steve Kaczkowski, Achim Kehrein (Germany), Murray S. Klamkin (Canada), Elias Lampakis (Greece), Kee-Wai Lau (China), Northwestern University Math Problem Solving Group, Albert D. Polimeni, Rob Pratt, Phillip P. Ray, Rolf Richberg (Germany), Joel Schlosberg, Harry Sedinger, Achilleas Sinefakopoulos, Nicholas C. Singer, John W. Spellmann, Nora Thornber, Dave Trautman, Chu Wenchang and Magli Pierluigi (Italy), Michael Vowe (Switzerland), John T. Zerger, Li Zhou, and the proposers.

Much Ado About Nothing **February 2003**

1663. *Proposed by Michel Bataille, Rouen, France.*

Let m and n be integers such that $1 \le m < n+1$. Evaluate

$$\sum_{k=1}^{n+1} \left((k+1) \sin^{k-1}\left(\frac{2\pi m}{n+1}\right) \prod_{j=1}^{k} \left(\cot\left(\frac{\pi m}{n+1}\right) - \cot\left(\frac{\pi j}{k+1}\right) \right) \right).$$

Solution by Chu Wenchang and Di Claudio Leontina Veliana, Universitàdegli Studi di Lecce, Lecce, Italy.

The sum is equal to 0. To prove this, we establish the more general result that for any real θ,

$$\sum_{k=1}^{n+1} (k+1) \sin^{k-1}(2\theta) \prod_{j=1}^{k} \left(\cot\theta - cot\frac{j\pi}{k+1} \right) = 2^{n+1} \frac{\sin(n+1)\theta}{\sin^2\theta} \cos^{n+1}\theta. \quad (1)$$

Setting $\theta = m\pi/(n+1)$, we see that the sum in the problem statement is 0.

Because

$$\sin((k+1)\theta) = \operatorname{Im}\left((\cos\theta + i\sin\theta)^{k+1}\right) = \sum_{0 \le j \le k/2} (-1)^j \binom{k+1}{2j+1} \sin^{2j+1}\theta \cos^{k-2j}\theta,$$