Mathematics Magazine

Problem 1637

Proposed by Erwin Just (Emeritus), Bronx Community College, Bronx, NY

Prove that the circle with equation $x^2 + y^2 = 1$ contains an infinite number of points with rational coordinates such that the distance between each pair of the points is irrational.

Solution:

For any $q \in \mathbb{Q}$, the point $(x,y) = P(q) = (\frac{1-q^2}{1+q^2}, \frac{2q}{1+q^2})$ is a rational point on the circle $x^2 + y^2 = 1$. Consider points where $q \in \mathbb{N}$. By direct calculation, the points $P(m)$ and $P(n)$ are separated by a distance $d(P(m), P(n)) = \sqrt{\left(\frac{1-m^2}{1+m^2} - \frac{1-n^2}{1+n^2}\right)^2 + \left(\frac{2m}{1+m^2} - \frac{2n}{1+n^2}\right)^2}$. This distance simplifies to $\frac{(2)(m-n)}{\sqrt{(1+m^2)(1+n^2)}}$, which is irrational unless $(1+m^2)(1+n^2)$ is a perfect square.

First, note that a number of the form $1 + n^2$ is never a perfect square itself. Moreover, the primes which appear in a factorization of $1 + n^2$ are necessarily different than the primes which appear in a factorization of n^2, hence n. To see this, note that if p is a prime such that $p|1 + n^2$ and $p|n^2$, then $p|1$, which is absurd.

Let $P_1 = P(1) = (0, 1)$. Now suppose we have a set of points $\{P_i = P(n_i) | 1 \leq i \leq k\}$ which satisfy the condition. Collectively, the primes which appear in the complete factorizations of the numbers $1 + n_i^2$ for $1 \leq i \leq k$ constitute a finite set, say $\{p_1, p_2, \cdots p_t\}$. Choose $n_{k+1} = \prod_{j=1}^{t} p_j$ so that $1 + n_{k+1}^2$ has no prime factor in common with n_{k+1}, hence no prime factor in common with any of the numbers $1 + n_i^2$ for $1 \leq i \leq k$, by our previous remark. As noted above, $1 + n_{k+1}^2$ is not a square, so the factorization of $1 + n_{k+1}^2$ must contain some prime to an odd power. But this implies that the product $(1 + n_i^2)(1 + n_{k+1}^2)$ for $1 \leq i \leq k$ has factorization containing that prime to an odd power, hence $(1 + n_i^2)(1 + n_{k+1}^2)$ is not a perfect square, and $P(n_{k+1})$ may be added to the set, establishing the result.

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12–10-01

Rational Points, Irrational Spacing **December 2001**

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Prove that the circle with equation $x^2 + y^2 = 1$ contains an infinite number of points with rational coordinates such that the distance between each pair of the points is irrational.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

Let p be a prime with $p \equiv 1 (\bmod 4)$. Then there are positive integers a and b with $p = a^2 + b^2$. The fractions $x_p = (a^2 - b^2)/p$ and $y_p = 2ab/p$ are in lowest terms and the point $P = (x_p, y_p)$ is on the unit circle. Let $q \neq p$ be an odd prime with $q = c^2 + d^2$ for some positive integers c and d, and let $Q = ((c^2 - d^2)/q, 2cd/q)$. Then $P \neq Q$ and

$$PQ = \frac{2|ad - bc|}{\sqrt{pq}}.$$

Because p and q are distinct primes, PQ is irrational. Because there are infinitely many primes that are congruent to 1 modulo 4, this construction provides a set of infinitely many points with rational coordinates on the unit circle and such that the distance between each pair of points in the set is irrational.

Also solved by Reza Akhlaghi, Michael Andreoli, Michel Bataille (France), Brian D. Beasley, Mihály Bencze (Romania), Alper Cay (Turkey), John Christopher, Charles R. Diminnie, Daniele Donini (Italy), Brenda Edmonds and Dale Hughes, Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), FGCU Problem Group, Ovidiu Furdui, Ken Korbin, Elias Lampakis (Greece), David Levitt, Peter A. Lindstrom, Kandasamy Muthuvel, Michael Reid, Ralph Rush, Ossama A. Saleh and Stan Byrd, Achilleas Sinefakopoulos, Michael Vowe (Switzerland), Rex H. Wu, Li Zhou, and the proposer. There were two incorrect submissions.

Answers

Solutions to the Quickies from page 400.

A925. Let D be the value of the determinant. Clearing fractions we have

$$xyz(y+z)(z+x)(x+y)D = \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (z+x)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}. \qquad (*)$$

Because the resulting determinant vanishes when $x = 0$ or $y = 0$ or $z = 0$, it has xyz as a factor. Next note that if $x + y + z = 0$, then the determinant has three proportional rows. Hence the determinant in $(*)$ also has $(x + y + z)^2$ as a factor. Thus the determinant in $(*)$ has the form

$$Pxyz(x + y + z)^2.$$

Because this determinant is a symmetric, homogeneous polynomial of degree 6, it follows that $P = k(x + y + z)$ for some constant k. To determine k, set $x = y = z = 1$. The determinant in $(*)$ then has value 54 and it follows that $k = 2$. We then find

$$D = \frac{2(x+y+z)^3}{(y+z)(z+x)(x+y)}.$$

A926. Let the vertices of $\triangle ABC$ be given in counter-clockwise order, let D be a point in the plane of the triangle, and let **A**, **B**, and **C**, respectively, be the vectors from D to