Mathematics Magazine

Problem 1633

Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY

A palindromic composition of a positive integer n is a palindromic finite sequence of positive integers whose sum is n. As examples, $1,2,2,1$ *and* $2,1,1,2$ *are different palindromic compositions of* 6*, and* $10,3,10$ *is a palindromic composition of* 23*. Find the number of palindromic compositions of the positive integer n.*

Solution:

The required formula for $N(n)$, the number of palindromic compositions of n, is $N(n) = 2^{\lfloor n/2 \rfloor}$.

A palindromic composition must have a center term (possibly the virtual term 0, which would not be explicitly written but only serve as a mirror) about which the sequence is symmetric. Let us call either partial sequence to the left or right of the center a wing. Given n and the center m, the terms in a wing must add to $\frac{1}{2}(n-m)$. We will enumerate $N(n)$, by letting m take all admissible values and counting the corresponding ordinary compositions for one of the resulting wings. It is clear that this process counts each palindromic composition once and only once.

The values of m are constrained by the odd or even parity of n. If n is even, then m must be even, otherwise the sum of terms in each wing would have different parity, which is impossible in view of symmetry. If n is odd, then m must be odd for the same reason. In any case, $n-m$ must be even, possibly zero.

Case 1: n is even. Possible values for m are $0,2,4,\cdots,n$. Corresponding values for the sum of terms in a wing are then $\frac{n}{2}, \frac{n}{2}-1,\cdots,\frac{n}{2}-k,\cdots,0$, where k runs from 0 to $\frac{n}{2}$. The subcase where $m=n$ results in a single palindrome (n), so setting that subcase aside we may consider values of k between 0 and $\frac{n}{2}-1$. Recalling that the number of r-compositions of a number s is $\binom{s-1}{r-1}$, we have the total number of compositions of $\frac{n}{2}-k$ is $\sum_{r=1}^{\frac{n}{2}-k}\binom{\frac{n}{2}-k-1}{r-1}$. Re-indexing this sum gives $\sum_{t=0}^{\frac{n}{2}-k-1}\binom{\frac{n}{2}-k-1}{t}$, which is well-known (via the Binomial Theorem) to be $2^{\frac{n}{2}-k-1}$. Summing over k gives $\sum_{k=0}^{\frac{n}{2}-1}2^{\frac{n}{2}-k-1}$, which upon reversing the order of summation yields $\sum_{q=0}^{\frac{n}{2}-1}2^q = 2^{\frac{n}{2}}-1$. Adding back the subcase consisting of the singleton palindrome (n) gives the final result $N(n) = 2^{\frac{n}{2}}$.

Case 2: n is odd. Consider the number of palindromic compositions of $n-1$, which by Case 1 is $2^{\frac{n-1}{2}}$. Now examine where the extra 1 can be placed in the palindromes for $n-1$. It cannot be assigned to a wing without violating symmetry, so it must be added to the center. There is an obvious bijection between palindromes with identical wings but center terms differing by 1, hence for odd n we have $N(n) = 2^{\frac{n-1}{2}}$.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL

In triangle ABC, let $a = BC$, $b = CA$, and $c = AB$. Prove that

$$\frac{b+c}{a^2}\cos A + \frac{c+a}{b^2}\cos B + \frac{a+b}{c^2}\cos C \geq \frac{9}{a+b+c}.$$

1661. *Proposed by Götz Trenkler, Dortmund, Germany.*

Let P and Q be two $n \times n$ complex orthogonal projectors, that is, $P = P^* = P^2$ and $Q = Q^* = Q^2$. Prove that PQ is an orthogonal projector if and only if PQ is normal, that is, if and only if $(PQ)^*PQ = PQ(PQ)^*$.

Quickies

Answers to the Quickies are on page 404.

Q925. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the determinant

$$\begin{vmatrix} \frac{y+z}{x} & \frac{x}{y+z} & \frac{x}{y+z} \\ \frac{y}{z+x} & \frac{z+x}{y} & \frac{y}{z+x} \\ \frac{z}{x+y} & \frac{z}{x+y} & \frac{x+y}{z} \end{vmatrix}.$$

Q926. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Find the maximum value of

$$\sin^2(2A) + \sin^2(2B) + \sin^2(2C) + 2\cos(2A)\sin(2B)\sin(2C)$$
$$+ 2\cos(2B)\sin(2C)\sin(2A) + 2\cos(2C)\sin(2A)\sin(2B),$$

where A, B, C are the angles of a triangle ABC.

Solutions

Palindromic Compositions **December 2001**

1633. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn NY.*

A palindromic composition of a positive integer n is a palindromic finite sequence of positive integers whose sum is n. As examples, 1, 2, 2, 1 and 2, 1, 1, 2 are different palindromic partitions of 6, and 10, 3, 10 is a palindromic partition of 23. Find the number of palindromic compositions of the positive integer n.

Solution by Malinda Roth, student, Westmont College, Santa Barbara, CA.

The number of palindromic compositions of the positive integer n is $2^{\lfloor n/2 \rfloor}$. If we represent the integer n as a string of n 1s, then each composition can be represented by inserting commas between some of the 1s. For example, the arrangement 1 1, 1, 1, 1, 1 1 represents the composition 2, 1, 1, 1, 2 of 7. Because the compositions are to be palindromic, any distribution of commas must be symmetric about the midpoint of the string of 1s. If n is odd, then there are $n - 1$ spaces between 1s, and any symmetric distribution of commas is completely determined by the placement of commas in the first $(n-1)/2 = \lfloor n/2 \rfloor$ spaces. There are $2^{\lfloor n/2 \rfloor}$ ways to select spaces for the commas. If n is even, then again there are $n-1$ spaces between 1s. A symmetric placement of commas is determined by the placement of commas in the first $(n-2)/2+1 = n/2 = \lfloor n/2 \rfloor$ spaces between 1s. As before, there are $2^{\lfloor n/2 \rfloor}$ different ways to select spaces for the commas.

Note: R. S. Tiberio of Wellesley, MA noted that this problem also appeared as Problem 1026 in the January 1979 issue of the MAGAZINE. See Vol. 53, No. 1, page 55.

Also solved by Michael Andreoli, The Assumption College Problems Group, Herb Bailey, Roy Barbara (Lebanon), Michel Bataille (France), J. C. Binz (Switzerland), Dorothee Blum, Jean Bogaert (Belgium), David W. Carter, Eddie Cheng, John Christopher, Con Amore Problem Group (Denmark), R. Flores Coombs (Chile), Robert DiSario, Daniele Donini (Italy), Mike Engling, Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), FGCU Problem Group, Ralph P. Grimaldi, Jerrold W. Grossman, Joel D. Haywood, Mack Hill, Brian Hogan, Heather Heston, Jerry G. Ianni, The Ithaca College Solvers, Gerald T. Kaminski and Robert L. Raymond, Koopa Tak-Lun Koo, Victor Y. Kutsenok, La Salle University Problem Solving Group, Kenneth Levasseur, David Levitt, Marvin Littman, Kevin McDougal, Perry and the Masons Solving Group (Spain), Rob Pratt, Leon Quet, Alex Rand, Joel Schlosberg, David Seff, Nicholas C. Singer, Skidmore College Problem Group, W. R. Smythe, David Treat, David Trautman, Michael Woltermann, Li Zhou, Seth Zimmerman, and the proposer. There was one incorrect submission.

A Constant for an Inequality **December 2001**

1634. *Proposed by Constantin P. Niculescu, University of Craiova, Craiova, Romania.*

Find the smallest constant $k > 0$ such that

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \le k(a+b+c)$$

for every $a, b, c > 0$.

A composite of solutions from Michel Bataille, Rouen, France; Daniele Donini, Bertinoro, Italy; Junhua Huang, Hunan Normal University, Changsha, China; and Li Zhou, Polk Community College, Winter Haven, FL.

The smallest such value of k is $1/4$. First note that for $x, y > 0$,

$$\frac{1}{x+y} = \frac{4xy}{x+y} \cdot \frac{1}{4xy} \le \frac{(x+y)^2}{x+y} \cdot \frac{1}{4xy} = \frac{1}{4}\left(\frac{1}{x} + \frac{1}{y}\right),$$

with equality if and only if $x = y$. We then have

$$\begin{aligned} &\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \\ &\quad \le \frac{ab}{4}\left(\frac{1}{a+c} + \frac{1}{b+c}\right) + \frac{bc}{4}\left(\frac{1}{c+a} + \frac{1}{b+a}\right) + \frac{ca}{4}\left(\frac{1}{c+b} + \frac{1}{a+b}\right) \\ &\quad = \frac{1}{4}(a+b+c), \end{aligned}$$

with equality if and only if $a+b = b+c = c+a$, that is, if and only if $a = b = c$.

Note: Howard Cary Morris of Cordova, TN investigated

$$f(a,b,c) = \left(\frac{1}{a+b+c}\right)\left(\frac{ab}{a+b+c+tc} + \frac{bc}{b+c+a+ta} + \frac{ca}{c+a+b+tb}\right)$$