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Problem 879

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Consider the polynomial $f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$ where a, b, c, and d are positive real numbers. Prove that if f has four positive distinct roots, then $a > b > c > d$.

Solution:

Let the zeroes of f be r_i, $1 \le i \le 4$.and we may assume $r_1 < r_2 < r_3 < r_4$ Evidently $d^4 = r_1r_2r_3r_4$, and $c^3 = \dfrac{r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4}{4}$. The arithmetic mean vs.geometric mean inequality applied to the products of the distinct zeroes in triples gives $c^3 > (r_1r_2r_3 \cdot r_1r_2r_4 \cdot r_1r_3r_4 \cdot r_2r_3r_4)^{1/4} = (r_1r_2r_3r_4)^{3/4} = d^3$, or $c > d$.

Consider $g(x) = \dfrac{1}{4}f'(x) = x^3 - 3ax^2 + 3b^2x - c^3$. The zeroes of g are the same as those of f', which occur at the three stationary points of f, located one each between the pairs of adjacent zeroes of f, and hence are necessarily positive and distinct. More specifically, denoting the zeroes of g by s_i, $1 \le i \le 3$, with $s_1 < s_2 < s_3$, we must have $r_1 < s_1 < r_2 < s_2 < r_3 < s_3 < r_4$. Repeating the previous argument now for g, since $c^3 = s_1s_2s_3$ and $b^2 = \dfrac{s_1s_2 + s_1s_3 + s_2s_3}{3}$, we have $b^2 > (s_1s_2 \cdot s_1s_3 \cdot s_2s_3)^{1/3} = (s_1s_2s_3)^{2/3} = c^2$, or $b > c$.

Finally, consider $h(x) = \dfrac{1}{3}g'(x) = x^2 - 2ax + b^2$. The zeroes of h are the same as those of g', which occur at the two stationary points of g, again located one each between the pairs of adjacent zeroes of g, ensuring that they are positive and distinct. Denoting the zeroes of h by t_1 and t_2, it follows that $b^2 = t_1t_2$ and $a = \dfrac{t_1 + t_2}{2}$, hence by the usual argument $a > \sqrt{t_1t_2} = b$, and we see that $a > b > c > d$. ■

We note that the above method would apply to the more general polynomial $\phi(x) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\alpha_i^i x^{n-i}$, where $\alpha_0 = 1$, $\alpha_i \in \mathbb{R}_+$, and the zeroes of ϕ are positive and distinct, and imply that $\alpha_i > \alpha_j$ if $0 < i < j$. For example, if $\phi(x) = x^5 - 5ax^4 + 10b^2x^3 - 10c^3x^2 + 5d^4x - e^5$, with the distinct zeroes r_i, $1 \le i \le 5$, we would have immediately that $d^4 = \dfrac{r_1r_2r_3r_4 + r_1r_2r_3r_5 + r_1r_2r_4r_5 + r_1r_3r_4r_5 + r_2r_3r_4r_5}{5}$ and $e^5 = r_1r_2r_3r_4r_5$, hence by the AM-GM inequality, $d^4 > (r_1r_2r_3r_4r_5)^{4/5} = e^4$, or $d > e$, and so forth.

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Roots and coefficients of a quartic function

879. *Proposed by Dorin Marghidanu, Colegiul National "A. I. Cuza," Corabia, Romania.*

Consider the polynomial $f(x) = x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4$, where a, b, c, and d are positive real numbers. Prove that if f has four positive distinct roots, then $a > b > c > d$.

Solution by Ronald Mosier, Ann Arbor, MI and José Nieto, Universidad del Zulia, Maracaibo, Venezuela (independently).

We shall use induction to prove the following generalization: If the polynomial

$$f(x) = x^n + \sum_{j=1}^{n} (-1)^j \binom{n}{j} u_j^j x^{n-j}$$

has n distinct positive roots, then $u_1 > u_2 > \cdots > u_n$.

For $n = 2$, $f(x) = x^2 - 2u_1x + u_2^2$. Let v_1 and v_2 be the two distinct positive roots of f. Then $u_1 = \frac{v_1+v_2}{2}$ and $u_2 = \sqrt{v_1v_2}$, and $u_1 > u_2$ by the Arithmetic-Geometric Mean Inequality. Assume that the desired result is true for $n = k - 1$. Consider

$$f(x) = x^k + \sum_{j=1}^{k} (-1)^j \binom{k}{j} u_j^j x^{k-j},$$

with k distinct positive roots $v_1, v_2, \ldots, v_k$. Then

$$\frac{f'(x)}{k} = x^{k-1} + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} u_j^j x^{k-1-j}.$$

By Rolle's Theorem, it has $k - 1$ distinct positive zeros. Hence, it follows from the induction hypothesis that $u_1 > u_2 > \cdots > u_{k-1}$, and it only remains to show that $u_{k-1} > u_k$. Since

$$\begin{aligned} u_{k-1}^{k-1} &= \frac{1}{k}(v_2v_3\cdots v_k + v_1v_3\cdots v_k + \cdots + v_1v_2\cdots v_{k-1}), \\ &> (v_1^{k-1}v_2^{k-1}\cdots v_k^{k-1})^{1/k} = u_k^{k-1}. \end{aligned}$$

Hence, $u_{k-1} > u_k$ and the proof is complete.

Remark. The positive real numbers a, b, c, and d of the problem are the symmetric means of the roots. The chain of inequalities for the symmetric means of n positive numbers was proved in 1729 by Maclaurin. (See Theorem 52, Hardy, Littlewood, and Polya, *Inequalities*, Cambridge, University Press.) The proof above is different from Maclaurin's.

Also solved by George Apostolopoulos, Messolonghi, Greece; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), Angelo State U.; Michel Bataille, Rouen, France; Tom Beatty, Florida Gulf Coast U.; Minh Can, Irvine Valley C.; Hongwei Chen, Christopher Newport U.; John Christopher, California State U., Sacramento; Margaret Cibes, Trinity C., Hartford, CT; Elliott Cohen, Fontenay-sous-Bois, France; Chip Curtis, Missouri Southern State U.; James Duemmel, Bellingham, WA; John Ferdinands, Calvin C.; Michael Goldenberg and Mark Kaplan (jointly), Baltimore Poly. Inst.; Hyun Sub Hwang, Inst. of Science Education for the Gifted and Talented, Yonsei U., Seoul, Korea; Eugen Ionascu,