College Mathematics Journal

Problem 748

Proposed by Kent Holing, Statoil Research Centre, Trondheim, Norway

Show that the two differently shaded regions in the figure below have the same area.
(See Drawing #1)

Solution: Restating the problem in terms of the solution drawing notation (see Drawing #2), we are to show that the square *OFGP* and the rectangle *PGHI* taken together have the same area as the combined areas of the rectangle *BCON*, the triangle *CFO*, and the triangle *AMJ*. The area of *OFGP* is a^2, and the area of *PGHI* is ab, since *ACED* is a square. By similar triangles, we have $|ON|/a = a/c$, hence $|ON| = a^2/c$. Then the area of the rectangle *BCON* is a^2. The altitude of triangle *CFO* to the base *OF* is $c\cos\theta$, that is to say b, hence the area of triangle *CFO* is $\frac{1}{2}ab$. The altitude of triangle *AMJ* to the base *MJ* is $c\sin\theta$, that is to say a, hence the area of triangle *AMJ* is $\frac{1}{2}ab$. Since $a^2 + ab = a^2 + \frac{1}{2}ab + \frac{1}{2}ab$, the areas are seen to be equal.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL
03-10-03

Thus,

$$S_2 - F_k^2 \geq \frac{(S_1 - F_k)^2}{n-1},$$

so that

$$\sum_{k=1}^{n} \frac{S_2 - F_k^2}{S_1 - F_k} \geq \sum_{k=1}^{n} \frac{S_1 - F_k}{n-1} = S_1 = F_{n+2} - 1.$$

Also solved by REZA AKHLAGHI, Prestonburg C.C.; TSEHAYE ANDEBRHAN, Asmara, Eritrea; MICHEL BATAILLE, Rouen, France; RICH BAUER, Shoreline, WA; JOHN CHRISTOPHER, California State U.-Sacramento; HABIB Y. FAR, Montgomery C.; OVIDIU FURDUI, Western Michigan U.; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; STEPHEN KACZKOWSKI, Orange County C.C.; DAVID E. MANES, SUNY C. at Oneonta; RONALD L. PERSKY, Christopher Newport U.; WILLIAM SEAMAN, Albright C.; MICHAEL VOWE, Therwil, Switzerland; LI ZHOU, Polk C. C.; and the proposer.

Shades of Pythagoras

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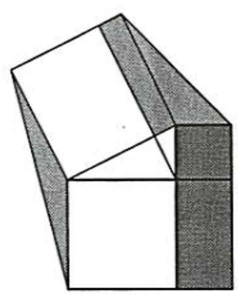
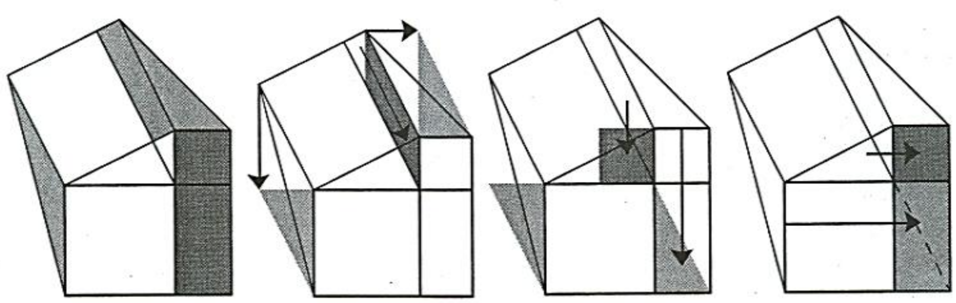
Show that the two differently shaded regions in the figure below have the same area.



Solution by Roger B. Nelsen, Lewis & Clark College, Portland, OR



Also solved by REZA AKHLAGHI, Prestonsburg C.C.; AYOUB B. AYOUB, Pennsylvania State U.-Abington C.; ROY BARBARA, American U. of Beirut; MICHEL BATAILLE, Rouen, France; KEVIN BERRY, California State U.-Northridge; PAUL BRACKEN, Concordia U., Montreal, Canada; MARK de SAINT-RAT, Miami U.; CHARLES R. DIMINNIE, Angelo State U.; BARBARA FALKOWSKI, California State U.-Northridge;

HABIB Y. FAR, Montgomery C.; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; OVIDIU FURDUI, Western Michigan U.; GEORGE WASHINGTON UNIVERSITY PROBLEMS GROUP; MICHAEL GOLDENBERG and MARK KAPLAN (jointly), Baltimore Polytechnic Institute; PETER HOHLER, Aarburg, Switzerland; RICKY IKEDA, Leeward C.C.; KHUDIJA S. JAMIL (student), California State U.-Northridge; STEPHEN KACZKOWSKI, Orange County C.C.; SEAN KELLER, Deerfield Academy, Deerfield, MA; DAVID E. MANES, SUNY C. at Oneonta; MAJID MASSO, George Mason U.; G. MAVRIGIAN, Youngstown, OH; JODIE MCCAULEY, Lipscomb U.; MICHAEL SCOTT MCCLENDON, U. of Central Oklahoma; IOANA MIHAILA, Cal Poly Pomona; NORTHWESTERN UNIVERSITY MATH PROBLEM SOLVING GROUP; JOHN POSCH, Greenville, CA; RICHARD C. RITTER, Lake Preston H.S., Lake Preston, SD; WILLIAM SEAMAN, Albright C.; HELEN SKALA, Winona, MN; H. T. TANG, Hayward, CA; NORA S. THORNBER, Raritan Valley C. C.; LINDA USELMANN, Edgewood C.; MICHAEL VOWE, Therwil, Switzerland; THOMAS C. WALES, Cambridge, MA; JOHN T. WARD, Marshall, MO; REX H. WU, Brooklyn, NY; YAJUN YANG, Farmingdale State U. of New York; LI ZHOU, Polk C.C.; and the proposer. One incorrect submission was received.

To fully diversified and beyond

749. *Proposed by Michael W. Ecker, Wilkes-Barre Campus, Pennsylvania State University, Lehman, PA*

For n a positive integer, let $N_n = \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ denote a finite sequence of subsets of N_n. Call this sequence "fully diversified" provided

(a) A_j has even order for $j = 1, 2, \ldots, k$

(b) For each element m in N_n there are exactly m values of j such that $m \in A_j$.

Find all values of n such that a fully diversified sequence exists.

Solution by S. Floyd Barger (Professor Emeritus), Youngstown State University, Youngstown, OH

More generally, if r is a positive integer, we call the sequence $A_1, A_2, \ldots, A_n$ r-fully diversified provided (a) each A_j has even order and (b) for each $m \in N_n$, there are exactly m^r values of j such that $m \in A_j$. We will show that an r-fully diversified sequence exists if and only if

(i) **Parity Condition:** n is congruent modulo 4 to either 0 or -1, and

(ii) **Non-Degeneracy Condition:** $n^r \leq 1^r + 2^r + 3^r + \cdots + (n-1)^r$.

(Necessity) Since each A_j has even order, the sum of their orders is even. If the sequence is r-fully diversified, then the sum will equal $1^r + 2^r + \cdots + (n-1)^r + n^r$, which has the same parity as $1 + 2 + \cdots + n = n(n+1)/2$. But $n(n+1)/2$ is even if and only if n is congruent to either 0 or -1 modulo 4. Thus, condition (i) must be satisfied. Furthermore, in an r-fully diversified sequence, each of the n^r terms containing n must contain at least one other of $1, 2, \ldots, n-1$. But then, (ii) must hold as well. [Note that for $r = 1$, (i) implies (ii).]

(Sufficiency) The following two-stage algorithm constructs the required sequence using two-element subsets.

Stage One (Greedy): Each $m \in N_n$ has a quota of m^r occurrences in terms of the sequence. At each step, select the two *largest* members b and c of N_n with unfilled quotas and include the set $\{b, c\}$ as the next term in the sequence. Iterate this step until either all quotas are filled, in which case stop, or until there is exactly one element a with unfilled quota, in which case go to Stage Two.

Before describing Stage Two, we make some observations about the outcome of Stage One. First of all, if all quotas are satisfied after Stage One, then the desired