College Mathematics Journal

Problem 735

Proposed by Norman Bolton, Birmingham, MI, and Jerrold W. Grossman, Oakland University, Rochester, MI

Persons $p_1, p_2, \cdots p_n$, $n > 1$ have been assigned seats $s_1, s_2, \cdots s_n$ on a tour bus. The passengers board in numerical order. Unable to follow directions, p_1 chooses a seat uniformly at random. Subsequent boarders take their assigned seat if available, and choose a vacant seat at random if not. Find the probability that p_n sits in s_n.

Solution: We examine all admissible configurations of n bus passengers boarding a bus with n seats. The probability that the n^{th} passenger will get his correct seat will be the ratio of the number of those configurations where he does get it divided by the total number of admissible configurations. Let i index the passengers and j index the seats. We have for $1 \leq i,j \leq n$ the seating function $f(i) = j$, which puts passenger i into seat j. Note that f is an injection, since nobody is sitting on anyone's lap. We shall call a passenger "displaced" if $f(i) \neq i$. Any configuration defines a set D, possibly void, of displaced passengers. Obviously D cannot be a singleton. Let the indices corresponding to displaced passengers be listed in increasing order as $D = \{i_1, i_2, \cdots i_k\}$. Several observations are immediate from the boarding rules. If $D \neq \emptyset$, then $i_1 = 1$. Also, $f(i_1) = i_2, f(i_2) = i_3, \cdots, f(i_{k-1}) = i_k$, and $f(i_k) = i_{k+1} = 1$. This condition can be seen by considering the fact that if displaced passenger number i_m takes a seat other than seat number i_{m+1}, then passenger number i_{m+1} would not be displaced, contrary to the definition of D. Note that this condition sets up a bijection between admissible configurations and possible sets D, with the void set being mapped to the configuration in which everyone finds his correct seat.

Now consider a configuration with corresponding nonvoid $D = \{i_1, i_2, \cdots i_k\}$, and suppose that $i_k < n$. This is a configuration where passenger n gets his assigned seat. Associate this configuration with the one having $D' = \{i_1, i_2, \cdots i_k, n\}$, where passenger n gets bumped and must take seat 1. The family of sets of type D and the family of sets of type D' evidently are disjoint and have the same cardinality. By our preceding remarks, we have accounted for all displacement sets except that of the form $D' = \{i_1, n\} = \{1, n\}$, which is associated with the configuration that puts passenger 1 in seat n and vice versa. Associate this configuration with the one having $D = \emptyset$, and we see that there is a bijection between all configurations where the n^{th} passenger gets his assigned seat and all those where he does not. We conclude that the probability of him getting the right seat is $\frac{1}{2}$.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL
09–24-02

Finding the right seat

735. *Proposed by Norman Bolton, Birmingham, MI and Jerrold W. Grossman, Oakland University, Rochester, MI*

Persons $p_1, p_2, \ldots, p_n$, $n > 1$ have been assigned seats $s_1, s_2, \ldots, s_n$ on a tour bus. The passengers board in numerical order. Unable to follow directions, p_1 chooses a seat (uniformly) at random. Subsequent boarders take their assigned seat if available, and choose a vacant seat at random if not. Find the probability that p_n sits in s_n.

Solution by Brad Jensen (student), Westmont College, Santa Barbara, CA

Let P_n be the probability that p_n will sit in s_n. We will use induction to show that $P_n = 1/2$ for $n > 1$. First, with $n = 2$, if p_1 chooses s_1, then p_2 has probability 1 of getting the correct seat. Otherwise, p_2 has probability 0 of getting the correct seat. Thus, $P_2 = \frac{1}{2}(1+0) = \frac{1}{2}$.

Now, suppose $n > 2$ and that $P_m = 1/2$ for $1 < m < n$. If p_1 chooses s_1, then p_n has probability 1 of sitting in s_n, and if p_1 chooses s_n, then p_n has probability 0 of sitting in s_n. If p_1 chooses s_k where $1 < k < n$, then p_i sits in s_i for $1 < i < k$, and p_k will be faced with the choice of seat s_1 or seats s_{k+1} through s_n. Since this is essentially the same situation as if p_k were p_1 with $n - k + 1$ passengers and seats, $P_n = \frac{1}{n}(1 + 0 + (n-2)\frac{1}{2}) = \frac{1}{2}$ which completes the induction.

Also solved by MICHAEL ANDREOLI, Miami Dade C.C.; HERB BAILEY, Rose-Hulman Institute of Technology; THOMAS BASS, Carson-Newman C.; MICHEL BATAILLE, Rouen, France; DAVID M. BLOOM, Hartsdale, NY; VIRGINIA BOLTON, Swainsboro, GA; PIERRE BORNSZTEIN, Pontoise, France; MARC BRODIE, C. of St. Benedict; DOUG CASHING, St. Bonaventure U.; JOHN CHRISTOPHER, California State U.-Sacramento; PHIL CLARKE, L.A. Valley C.; CON AMORE PROBLEM GROUP, The Danish U. of Education, Copenhagen, Denmark; CHIP CURTIS, Missouri Southern State C.; KEITH EKBLAW, Walla Walla, WA; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; TOM GETTYS, Eugene, OR; THE HOFSTRA UNIVERSITY PROBLEM SOLVERS; PETER HOHLER, Aarburg, Switzerland; D. KIPP JOHNSON, Valley Catholic School, Beaverton, OR; STEPHEN KACZKOWSKI, Orange County C.C.; BRANDON KARLSGODT (student), Dordt C.; N. J. KUENZI, U. of Wisconsin-Oshkosh; JOY LAMBERT and ANDREW WHITE (jointly), Alfred U.; KEE-WAI LAU, Hong Kong, China; RICHARD F. MCCOART, JR., Loyola C.; MCDANIEL COLLEGE PROBLEMS GROUP; KEVIN MCDOUGAL, U. of Wisconsin-Oshkosh; SEAN MCILROY, Vancouver, Canada; DOUG MERKLE, Panama City, FL; LAURENCE D. MERKLE, Rose-Hulman Institute of Technology; ELIZABETH MILLER (student) and FARLEY MAWYER (jointly), York C. (CUNY); DARRYL K. NESTER, Bluffton C.; ROGER PINKHAM, Stevens Institute of Technology; ROB PRATT (student), UNC-Chapel Hill; ROBERT C. RHOADES (student), Bucknell U.; DON ST. JEAN, George Brown C., Toronto, Canada; JOSHUA SANDERS, The Evergreen State C.; WILLIAM SEAMAN, Albright C.; HARRY SEDINGER, St. Bonaventure U.; TAKAKO SHIMOHORI, Sendai, Japan; SKIDMORE COLLEGE PROBLEM GROUP; JOHN HENRY STEELMAN, Indiana U. of Pennsylvania; AMANDA L. STEPHENS, Rose-Hulman Institute of Technology; WILLIAM TRESSLER, Charter School of Wilmington, DE; UNIVERSITY OF ARIZONA PROBLEM SOLVING GROUP; LI ZHOU, Polk C.C.; SETH ZIMMERMAN, Evergreen Valley C.; and the proposer.

Editors' Note: The proposers noted that the problem was not original with them but that they had not seen the problem in print. In fact, the Con Amore Problem Group pointed out that the problem, with a different setting, appeared in the December 2001, (vol 15, no 2) issue of the journal *FAMØS* which is published by the University of Copenhagen. In the next two issues of *FAMØS*, March and May of 2002, a generalization of the problem was first proposed and then solved. The generalization asked for the probability that p_k will sit in s_k for $1 < k \le n$. As a number of the solvers listed above showed, this probability is

$$\frac{n-k+1}{n-k+2}$$

with $k = n$ solving the original problem. This shows that with the exception of p_1, p_n is the passenger who is least likely to end up in the correct seat.