College Mathematics Journal

Problem 731

Proposed by Michael Scott McClendon, University of Central Oklahoma, Edmond, OK

A spherical planet with radius R has a single airplane attached to a fixed point on the planet by an unbreakable rope of length $2\pi R$. If the airplane can fly anywhere allowed by the rope, what is the total volume of airspace?

Solution: To simplify the calculation, assume that the planet has radius 1. We will scale the problem with the factor R^3 once the volume for this case is determined. The airspace available to the plane has axial symmetry about the diameter of the planet that passes thru the tether point. Pass a plane (geometric, not air-) thru the planet containing this diameter and impose a cartesian coordinate system with the origin at the center of the planet and the tether attachment at $(1,0)$. The airspace consists of two zones: (i) the hemispherical zone of radius 2π that corresponds to the end of the tether in the half-space $x \geq 1$, and (ii) the solid of revolution consisting of rotating about the x-axis that portion of the involute created by the tether wrapping around the planet whenever $x < 1$ and $y \geq 0$, minus, of course, the volume of the planet itself. We note that the airplane can go either clockwise or counterclockwise around the planet, and once the extended diameter is crossed, the tether is shorter than if the path were from the opposite direction.

The parametric equations for the airplane's position in the involute zone are $x(t) = \cos t - (2\pi - t)\sin t$ and $y(t) = \sin t + (2\pi - t)\cos t$, where t is the usual polar angle referred to the positive x-axis. Referring to the figure, the angle t_{crit} is such that the involute described by the end of the tether meets the extended diameter. We see that t_{crit} satisfies the equation $\frac{1}{2\pi - t_{crit}} = \tan(t_{crit} - \frac{\pi}{2})$. Numerical solution of this transcendental equation gives $t_{crit} \simeq. 1.7898$. The volume of the solid of revolution determined by the involute is then $V_{involute} = \int_{1.7898}^{0} \pi[y(t)]^2 \frac{dx}{dt}\,dt$, which from the parametric equation is $\pi \int_{1.7898}^{0} [\sin t + (2\pi - t)\cos t]^2 \frac{d}{dt}[\cos t - (2\pi - t)\sin t]dt = 500.4116$. The volume of the hemispherical zone is $V_{hemi} = \frac{2}{3}\pi(2\pi)^3 = 519.5200$ and the volume of the planet is $V_{planet} = \frac{4}{3}\pi(1)^3 = 4.1888$. The airspace is finally $V_{hemi} + V_{involute} - V_{planet} = 519.5200 + 500.4116 - 4.1888 = 1015.7428$ units. Restoring the scaling factor R^3, the airspace for a tether of length $2\pi R$ attached to a planet of radius R is then $1015.7428R^3$ units.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL
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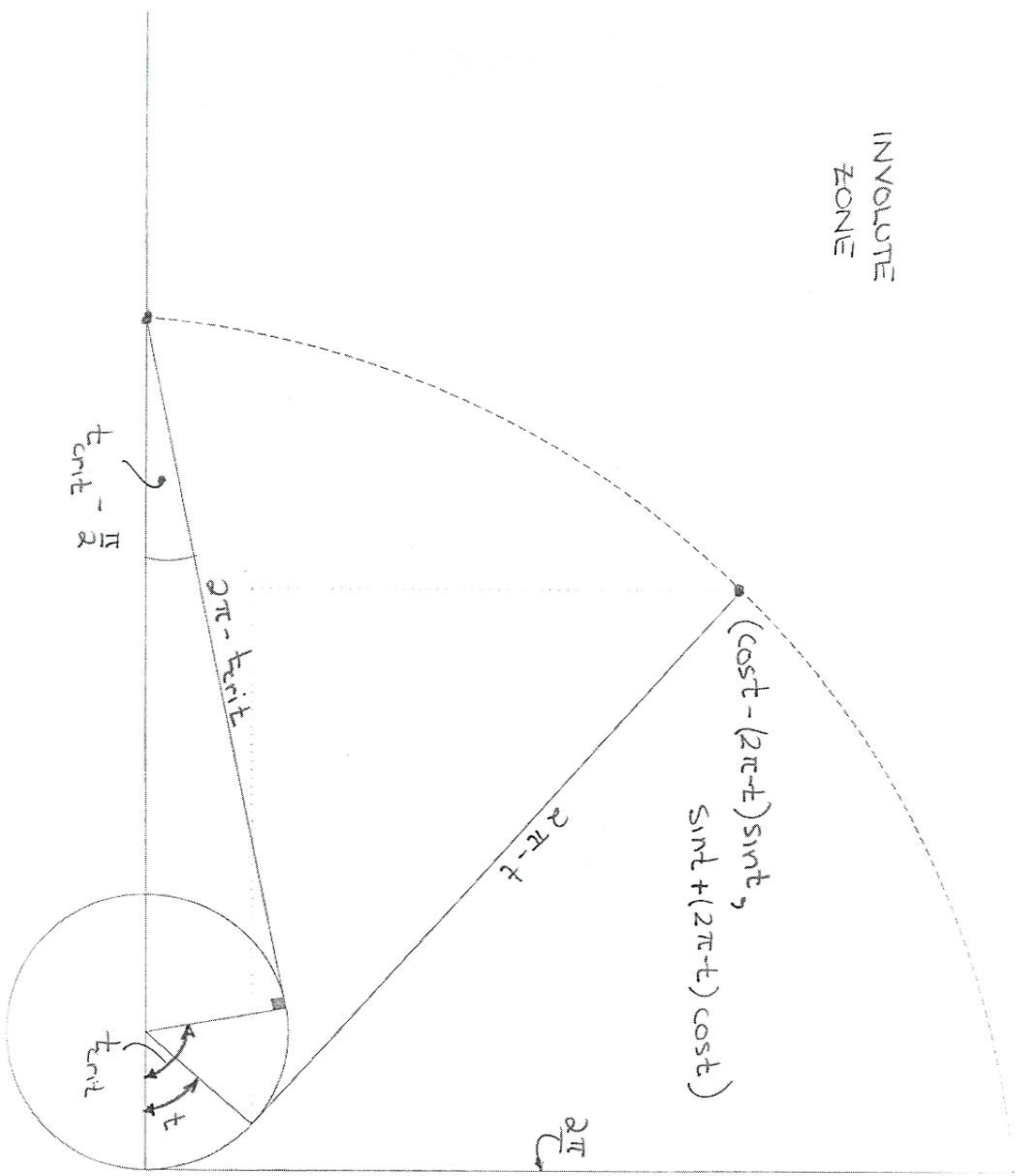

DWG NUMBER: DATE: REVISED:	CONTENTS: **TETHERED AIRPLANE**	
DRAWN BY: TOM CHECKED BY: SCALE: NONE	FOR THE PROJECT: **PROBLEM SEMINAR**	

759. *Proposed by Götz Trenkler, University of Dortmund, Germany*

If $\mathbf{A}$ and $\mathbf{B}$ are n by n matrices over an arbitrary field F, define $\mathbf{A} \circ \mathbf{B}$ to be the matrix $\mathbf{A} + \mathbf{B} - \mathbf{AB}$. Find necessary and sufficient conditions on $\mathbf{A}$ such that the equation $\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A} = \mathbf{0}$ has a solution, $\mathbf{B}$.

760. *Proposed by Arthur L. Holshouser, Charlotte, NC*

Suppose that f is a given function from the positive integers to the non-negative integers. We define a function g, whose domain is the non-negative integers, as follows: $g(0) = \infty$ and for n a positive integer, $g(n)$ is defined recursively by

$$g(n) \text{ is the smallest } x \text{ in } \{1, 2, \ldots, n\} \text{ such that } f(n) < g(n - x).$$

Note that $g(1) = 1$.

(a) If $f(n)$ is the largest power of 2 that divides n, find $g(n)$.

(b) Prove that for any f, if n is a positive integer and $1 \le x \le g(n) - 1$, then $g(n - x) \le g(n) - x$.

SOLUTIONS

The airspace of a tethered airplane

731. *Proposed by Michael Scott McClendon, University of Central Oklahoma, Edmond, OK*

A spherical planet with radius R has a single airplane attached to a fixed point on the planet by an unbreakable rope of length $2\pi R$. If the airplane can fly anywhere allowed by the rope, what is the total volume of its airspace?

Solution by Graig Simmonette (student), Penn State Wilkes-Barre, Wilkes Barre, PA

We first examine the lower portion of the airspace, ignoring for now the hemisphere on top of the airspace. In the figure the point L corresponds to the position, $P = P(\theta) = (x(\theta), y(\theta))$, of the plane when $x(\theta) = 0$.

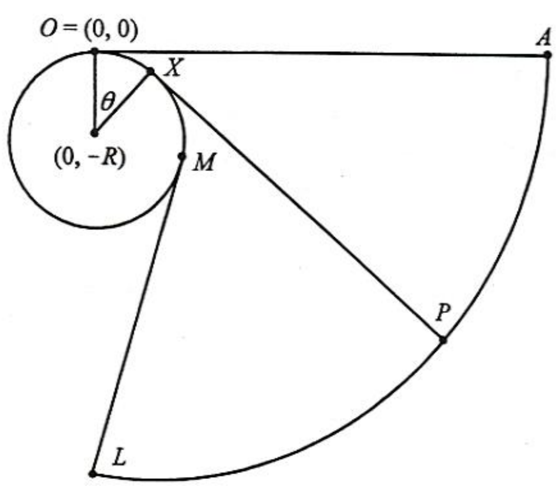

Figure 1.

From the figure we see that vectors OX and XP are given by

$$OX = R\langle \sin(\theta), \cos(\theta) - 1 \rangle \quad \text{and} \quad XP = R\langle (2\pi - \theta)\cos(\theta), -(2\pi - \theta)\sin(\theta) \rangle.$$

Thus, point P has coordinates

$$R\big(\sin(\theta) + (2\pi - \theta)\cos(\theta), \cos(\theta) - 1 - (2\pi - \theta)\sin(\theta) \big).$$

To find the volume of the lower portion of the airspace, we find the volume generated when the region bounded by segment OA, arc AL, and segment LO is revolved about the y-axis. We see that $x(\theta) = 0$ implies that $\tan(\theta) + 2\pi - \theta = 0$, and the solution to this equation is $\theta = \theta_0 = 1.7898$, accurate to four decimal places. Using cylindrical shells, the volume of the lower portion of the airspace is

$$2\pi \int_0^{\theta_0} x(\theta) y(\theta) \frac{dx}{d\theta}\, d\theta \approx 500.41 R^3.$$

To get the volume of the entire airspace, we subtract the volume of the planet from the volume of the lower portion given above and add the volume of the upper portion of the airspace, a hemisphere of radius $2\pi R$. We find the volume of the airspace is approximately $1015.74 R^3$.

Also solved by HERB BAILEY, Rose-Hulman Institute of Technology; MEGHAN BUTLER (student), Westmont C.; CHIP CURTIS, Missouri Southern State C.; JULIO G. DIX and JOHN SPELLMANN (jointly), Southwest Texas State U.; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; D. KIPP JOHNSON, Valley Catholic School, Beaverton, OR; STEPHEN KACZKOWSKI, Orange County C.C.; OSSAMA A. SALEH and STAN BYRD (jointly), U. of Tennessee at Chattanooga; WILLIAM SEAMAN, Albright C.; DANIEL SCHULTZ and JIM WEIR (students, jointly), Lamar U.; LI ZHOU, Polk C.C.; and the proposer. Three solutions contained small numerical errors. One submission made an interpretation deemed incorrect. One incorrect submission was received.

Not quite an integer

732. *Proposed by Ron Rietz, Gustavus Adolphus College, Saint Peter, MN*

(a) Find the smallest constant K such that for all $x \geq 1$, we have

$$0 \leq x^{\frac{1}{x}} - 1 - \frac{\ln(x)}{x} \leq K \left(\frac{\ln(x)}{x} \right)^2.$$

(b) Define

$$f(x) = \exp\left(x \frac{\exp(x) + 1}{\exp(x)} \right) - \exp(x).$$

If $f(1000) = N.d_1 d_2 d_3 ...$ where N is the integer part of $f(1000)$ and $0.d_1 d_2 d_3 ...$ is the decimal expansion of the fractional part of $f(1000)$, determine the value of $d_1 + d_2 + \cdots + d_{400}$.

Solution by Julien Grivaux (student), Université Pierre et Marie Curie, Paris, France

(a) Let $y = \frac{\ln(x)}{x}$. If $x \geq 1$, then $0 \leq y \leq \frac{1}{e}$. Define a function $g(t)$ for $t > 0$ by

$$g(t) = \frac{e^t - 1 - t}{t^2} = \sum_{n=2}^{\infty} \frac{t^{n-2}}{n!}.$$