College Mathematics Journal

Problem 726

Proposed by Thomas J. Pfaff, Ithaca College, Ithaca, NY

Find the value of $\lim_{n\to\infty}(\frac{1}{n^2}\sum_{k=1}^{n}\csc(\frac{1}{k}))$ *or prove that the limit does not exist.*

Solution:

To show that this limit is $\frac{1}{2}$, note that (from the Maclaurin expansion assuming $x \in (0,1]$) $x > \sin x > x - x^3/6 > 0$. Set $x = \frac{1}{k}$ and invert to obtain $k < \csc(\frac{1}{k}) < k(1-\frac{1}{6k^2})^{-1}$. Then $\sum_{k=1}^{n} k < \sum_{k=1}^{n}\csc(\frac{1}{k}) < \sum_{k=1}^{n} k(1-\frac{1}{6k^2})^{-1}$. Now choose an arbitrary index $m < n$ and rewrite the right hand sum as $\sum_{k=1}^{m} k(1-\frac{1}{6k^2})^{-1} + \sum_{k=m+1}^{n} k(1-\frac{1}{6k^2})^{-1}$.

Clearly $\sum_{k=m+1}^{n} k(1-\frac{1}{6k^2})^{-1} \le (1-\frac{1}{6m^2})^{-1}\sum_{k=m+1}^{n} k$, so we may assemble our partial results as follows:

$$\sum_{k=1}^{n} k < \sum_{k=1}^{n}\csc(\tfrac{1}{k}) < \sum_{k=1}^{m} k(1-\tfrac{1}{6k^2})^{-1} + (1-\tfrac{1}{6m^2})^{-1}\sum_{k=m+1}^{n} k.$$

Writing S_m for $\sum_{k=1}^{m} k(1-\frac{1}{6k^2})^{-1}$, and noting that $\sum_{k=m+1}^{n} k \le \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, we obtain $\frac{n(n+1)}{2} < \sum_{k=1}^{n}\csc(\frac{1}{k}) < S_m + (1-\frac{1}{6m^2})^{-1}(\frac{n(n+1)}{2})$.

It follows that:

$$\lim_{n\to\infty}(\tfrac{1}{n^2}\tfrac{n(n+1)}{2}) \le \lim_{n\to\infty}(\tfrac{1}{n^2}\sum_{k=1}^{n}\csc(\tfrac{1}{k})) \le \lim_{n\to\infty}(\tfrac{1}{n^2}S_m) + \lim_{n\to\infty}(\tfrac{1}{n^2}(1-\tfrac{1}{6m^2})^{-1}(\tfrac{n(n+1)}{2}))$$

and since $\lim_{n\to\infty}(\frac{1}{n^2}S_m) = 0$, we have $\frac{1}{2} \le \lim_{n\to\infty}(\frac{1}{n^2}\sum_{k=1}^{n}\csc(\frac{1}{k})) \le \frac{1}{2}(1-\frac{1}{6m^2})^{-1}$.

But m was arbitrary, so $\lim_{n\to\infty}(\frac{1}{n^2}\sum_{k=1}^{n}\csc(\frac{1}{k})) = \frac{1}{2}$.

FGCU Problem Group
Florida Gulf Coast University
Ft Myers, FL
05-15-02

754. *Proposed by Andrew Cusumano, Great Neck, NY*

(a) Prove that if p is a non-negative integer, there are positive integers $a_{p,m}$ with $1 \le m \le p+1$ such that

$$\sum_{k=1}^{n} k^p = \sum_{m=1}^{p+1} a_{p,m} \binom{n}{m}$$

and, if we define $a_{p,m} = 0$ for $m < 1$ or $m > p+1$, find a linear recurrence for $a_{p+1,m}$ in terms of $a_{p,j}$ with $j \le m$.

(b) For $p > 0$, find the value of $\sum_{m=1}^{p+1} a_{p,m}(-1)^m$.

755. *Proposed by Dean Hoffman and Peter Johnson, Auburn University, Auburn, AL*

Let $I_n = \{1, 2, \ldots, n\}$ be the set of outcomes of a probabilistic experiment in which the probability that outcome x occurs is $\pi(x)$. The set I_n is partitioned into k non-empty events, $E_1, \ldots, E_k$. A gambler, who knows the value of the density function π, is informed which event occurred and then chooses an outcome in that event. If his guess is correct, he receives a payoff of $1/\pi(x)$; otherwise he receives nothing. A "strategy" for the gambler is a set of density functions $\gamma_i(x)$, one for each of the events E_i. Thus if the gambler is informed that event E_i occurred, he chooses outcome x in E_i with probability $\gamma_i(x)$.

(a) What strategy should the gambler employ if he wishes to *maximize* the expected value of the payoff he receives? What is the associated expected value?

(b) What strategy should the gambler employ if he wishes to *minimize* the expected value of the payoff he receives? What is the associated expected value?

SOLUTIONS

A limit involving the $x/\sin(x)$ approximation

726. *Proposed by Thomas J. Pfaff, Ithaca College, Ithaca, NY*

Find the value of

$$\lim_{n\to\infty} \left(\frac{1}{n^2} \sum_{k=1}^{n} \csc\left(\frac{1}{k}\right) \right)$$

or prove that the limit does not exist.

Solution by Jim Hartman, The College of Wooster, Wooster, OH

The value of the limit is $\frac{1}{2}$. For $0 < x < \frac{\pi}{2}$, we know that

$$\frac{6x - x^3}{6} = x - \frac{x^3}{6} \le \sin(x) \le x.$$

This implies that if k is a positive integer, then

$$k \le \csc\left(\frac{1}{k}\right) \le \frac{6}{\frac{6}{k} - \frac{1}{k^3}} = k + \frac{k}{6k^2 - 1} \le k + \frac{1}{5k}.$$

Thus,

$$\frac{n(n+1)}{2} \leq \sum_{k=1}^{n} \csc\left(\frac{1}{k}\right) \leq \frac{n(n+1)}{2} + \frac{1+\ln(n)}{5}.$$

Dividing by n^2 and taking the limit as $n \to \infty$ gives the desired result.

Also solved by MICHAEL H. ANDREOLI, Miami-Dade C. C. (North); CARMEN ARTINO, The C. of Saint Rose; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; MARC BRODIE, C. of St. Benedict; MARGARET CIBES, Hillyer C. of U. of Hartford; PHIL CLARKE, Los Angeles Valley C.; JOSÉ LUIS DÍAZ-BARRERO and JUAN JOSÉ EGOZCUE (jointly), U. Politècnica de Catalunya, Barcelona, Spain; DANIELE DONINI, Bertinoro, Italy; ROBERT DOWNES, Mountain Lakes H. S., Mountain Lakes, NJ; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; OVIDIU FURDUI, Western Michigan U.; MICHELLE GHRIST, U. S. Air Force Academy; BRIAN D. GINSBERG (student), Yale U.; TOMMY GOEBELER, Boothwyn, PA; JOHN GRAHAM, Penn State-Wilkes-Barre; JULIEN GRIVAUX, Paris, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; HERMAN HOU, Atlanta, GA; RICKY IKEDA, Leeward C.C.; PETER M. JARVIS, Georgia C. & State U.; STEPHEN KACZKOWSKI, Orange County C.C.; STEVE KIFOWIT, Prairie State C.; TOM KIMBER, SUNY-Morrisville; MURRAY S. KLAMKIN, U. of Alberta; KEE-WAI LAU, Hong Kong, China; JEROLD LEWANDOWSKI, West Springfield H.S.; JAMES MAGLIANO, Union County C.; THOMAS C. MCMILLAN and XIU YE (jointly), U. of Arkansas-Little Rock; LUIS F. MORENO, Broome C. C.; STEPHEN NOLTIE, Ohio U.-Lancaster; PHILIP OPPENHEIMER, Norwalk, CT; ROBERT POODIACK, Norwich U.; BILL REID, C.C. of Philadelphia; SUBAS SAHA (student) and FARLEY MAWYER (jointly), York C. (CUNY); R. P. SEALY, Mount Allison U.; WILLIAM SEAMAN, Albright C.; JOHN W. SPELLMANN, Southwest Texas State U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; MICHAEL VOWE, Therwil, Switzerland; THOMAS WALLGREN, Southeast Missouri State U.; HANSRUEDI WIDMER, Nussbaumen, Switzerland; ROB WILLIAMS, Alfred U.; LI ZHOU, Polk C.C.; and the proposer. Two incorrect solutions were received.

Rational function inequalities

727. *Proposed by Yves Nievergelt, Eastern Washington University, Cheney, WA*

For each pair of real numbers r and v such that $0 \leq r$ and $0 < v < 1$, rank the following four expressions in increasing order, noting all ties.

$$1-(1-v)^r,\ (1+v)^r-1,\ \frac{1}{(1-v)^r}-1,\ 1-\frac{1}{(1+v)^r}$$

Solution by Ricky Ikeda, Leeward Community College, Pearl City, HI

Let $A = 1-(1-v)^r$, $B = (1+v)^r - 1$, $C = \frac{1}{(1-v)^r} - 1$, and $D = 1 - \frac{1}{(1+v)^r}$. That $A = B = C = D = 0$ if and only if $r = 0$ is trivial, so suppose $r > 0$. Now $A - C = A - \frac{1}{1-A} + 1 = \frac{-A^2}{1-A} < 0$ since $0 < A < 1$. Also, $D - A = \frac{(1-v^2)^r - 1}{(1+v)^r} < 0$ since $0 < 1 - v^2 < 1$. Hence $D < A < C$. Similarly, $D < B < C$.

Let $f(v) = B - A = (1+v)^r + (1-v)^r - 2$. Then $f(0) = 0$ and $f'(v) = r[(1+v)^{r-1} - (1-v)^{r-1}]$. For $r = 1$, $f'(v) = 0$. For $0 < r < 1$, $f'(v) < 0$ if $0 < r < 1$ and $f'(v) > 0$ if $r > 1$. So we see that $D < B < A < C$ if $0 < r < 1$, $D < A = B < C$ if $r = 1$, and $D < A < B < C$ if $r > 1$.

Also solved by HERB BAILEY, Rose-Hulman Institute of Technology; MICHEL BATAILLE, Rouen, France; JOHN CHRISTOPHER, California State U.-Sacramento; MARGARET CIBES, Hillyer C. of U. of Hartford; PHIL CLARKE, Los Angeles Valley C.; HABIB Y. FAR, Montgomery C.; OVIDIU FURDUI, Western Michigan U.; MICHELLE GHRIST, U. S. Air Force Academy; BRIAN D. GINSBERG (student), Yale U.; JOHN GRAHAM, Penn State-Wilkes-Barre; RAYMOND N. GREENWELL, Hofstra U.; JULIEN GRIVAUX, Paris, France; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; STEPHEN