# College Mathematics Journal

**Problem 709**

*Proposed by Vern E. Heeren, American River College, Sacramento, CA*

*(a) Characterize the rational numbers q such that the equation* $\frac{\pi}{4} = \arctan q + \arctan x$ *has a rational solution x.*

*(b) Characterize the rational numbers q such that the equation* $\frac{\pi}{4} = \arctan q + 2\arctan x$ *has a rational solution x.*

**Solution:**

(a) Let $u = \arctan q$ and $v = \arctan x$. Then

$\tan(\frac{\pi}{4}) = 1 = \tan(\arctan q + \arctan x) = \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{q+x}{1-qx}$. It follows that $q = \frac{1-x}{1+x}$, with $x \neq -1$. Writing $x = \frac{m}{n}$, we have $q = \frac{n-m}{n+m} \in \mathbb{Q}$, with $m \in \mathbb{Z}, n \in \mathbb{Z} - \{0\}$, $(m,n) = 1$, and $m \neq -n$ (this is not redundant, as $m = 1$ and $n = -1$ would not be excluded by the GCD condition).

(b) Let $w = 2\arctan x = \arctan y$. Then $\tan w = \tan(2\arctan x) = \frac{2x}{1-x^2} = y$. From the preceding result, we have $q = \frac{1-y}{1+y} = \frac{x^2+2x-1}{x^2-2x-1}$, where the denominator can never vanish for $x \in \mathbb{Q}$. Writing $x = \frac{m}{n}$ with the same restrictions as above except $m \neq \pm n$, we have $q = \frac{m^2+2mn-n^2}{m^2-2mn-n^2} \in \mathbb{Q}$.

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## Rational solutions to Arctangent equations

**709.** *Proposed by Vern E. Heeren, American River College, Sacramento, CA*

(a) Characterize the rational numbers $q$ such that the equation

$$\frac{\pi}{4} = \arctan(q) + \arctan(x)$$

has a rational solution $x$.

(b) Characterize the rational numbers $q$ such that the equation

$$\frac{\pi}{4} = \arctan(q) + 2 \cdot \arctan(x)$$

has a rational solution $x$.

*Composite of Solutions by Michael Andreoli, Miami-Dade Community College, Miami, FL and William Seaman, Albright College, Reading, PA*

The answer to (a) is the set of all rational numbers $q > -1$. The answer to (b) is the set of all rational numbers $q$ such that $2 + 2q^2$ is the square of a rational number. Equivalently, if we write $q = a/b$, then $2(a^2 + b^2)$ must be a perfect square. The set of such rational numbers may also be characterized as the set of all rational numbers of the form $q = (u + v)/(u - v)$ where $|u|$ and $|v|$ are the legs of a Pythagorean triple. (Of course, the well-known parametrization of Pythagorean triples then yields a parametrization of the set of all such rational numbers $q$, if we allow one of $u$ and $v$ to be zero.)

(a) The equation

$$\frac{\pi}{4} = \arctan(q) + \arctan(x) \tag{1}$$

has no solution for $q \leq -1$, since this would imply that

$$\arctan(x) = \frac{\pi}{4} - \arctan(q) \geq \frac{\pi}{2}.$$

On the other hand, for $q > -1$, we have $-\frac{\pi}{4} < \frac{\pi}{4} - \arctan(q) < \frac{\pi}{2}$ and we see from the addition formula for tangent that a unique solution to equation (1) is obtained by taking

$$x = \tan\left[\frac{\pi}{4} - \arctan(q)\right] = \frac{1-q}{1+q}.$$

Clearly, if $q$ is rational then so is $x$, which completes the proof of part (a).

(b) The equation

$$\frac{\pi}{4} = \arctan(q) + 2\arctan(x) \tag{2}$$

has a solution if and only if there is a real number $x$ such that

$$\arctan(x) = \frac{1}{2}\left[\frac{\pi}{4} - \arctan(q)\right].$$

However, for any real number $q$ we have

$$-\frac{\pi}{8} < \frac{1}{2}\left[\frac{\pi}{4} - \arctan(q)\right] < \frac{3\pi}{8}$$

and a unique solution to equation (2) is obtained by taking

$$x = \tan\left(\frac{1}{2}\left[\frac{\pi}{4} - \arctan(q)\right]\right).$$

If $q = -1$, then the solution to equation (2) is $x = 1$. If $q \neq -1$, it follows from the addition formula for tangent that if $x$ and $q$ satisfy equation (2), then

$$\frac{2x}{1-x^2} = \tan(2\arctan(x)) = \tan\left(\frac{\pi}{4} - \arctan(q)\right) = \frac{1-q}{1+q}.$$

If $q = 1$, then the solution to equation (2) is $x = 0$, while for $|q| \neq 1$, it follows that the solution $x$ to (2) must also satisfy the quadratic equation

$$(1-q)x^2 + 2(1+q)x - (1-q) = 0.$$

Using this equation, the quadratic formula, and our examination of the cases $|q| = 1$, we conclude that $x$ is rational if and only if $2 + 2q^2$ is the square of a rational number. Equivalently, if $q = a/b$, then $x$ is rational if and only if $2(a^2 + b^2) = (a+b)^2 + (a-b)^2$ is a perfect square. Setting $u = a + b$ and $v = a - b$, it follows that $q = (u+v)/(u-v)$ where $|u|$ and $|v|$ are the legs of a Pythagorean triple. Conversely, if $u$ and $v$ are integers such that $|u|$ and $|v|$ are the legs of a Pythagorean triple and $q$ is defined by the formula $q = (u+v)/(u-v)$, then it is easy to check that $2 + 2q^2$ is the square of a rational number. This completes the proof of (b).

*Also solved by* HERB BAILEY and JOHN RICKERT (jointly), Rose-Hulman Institute of Technology; MICHEL BATAILLE, Rouen, France; JOHN CHRISTOPHER, California State U.-Sacramento; PHIL CLARKE, Los Angeles Valley C.; CHARLES K. COOK, U. of South Carolina-Sumter; DANIELE DONINI, Bertinoro, Italy; RAYMOND N. GREENWELL, Hofstra U.; DALE HUGHES and JEFF LEWIS (jointly), Johnson County C. C.; RICKY IKEDA, Leeward C. C.; STEPHEN KACZKOWSKI, Orange County C.C.; DARRYL K. NESTER, Bluffton C.; LI ZHOU, Polk C. C.; and the proposer. Solutions to part (b) were received from REZA AKHLAGHI, Prestonsburg C.C. and FARY SAMI, Harford C. C. (jointly); FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; OVIDUI FURDUI, Western Michigan U.; NATALIO H. GUERSENZVAIG, Universidad CAECE, Argentina; KEN KORBIN, New York, NY; SAM NORTHSHIELD, SUNY-Plattsburgh; JAY NYZOWYJ, Mt. Pleasant, MI; and ALEXEY VOROBYOV, Irvine, CA. One incorrect solution was received

## A generalized centroid

**710.** *Proposed by Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN*

Given a triangle $ABC$, choose $A'$ on $CB$, $B'$ on $AC$ and $C'$ on $BA$ such that

$$|CA'| : |CB| = |BC'| : |BA| = |AB'| : |AC|.$$

Let $P$ be the point in which $AA'$ and $BB'$ meet, let $Q$ be the point in which $BB'$ and $CC'$ meet, and let $R$ be the point in which $AA'$ and $CC'$ meet. Determine the locus of the centroid of the (possibly degenerate) triangle $PQR$.