College Mathematics Journal

Problem 707

Proposed by Dennis Walsh, Middle Tennessee State University, Murfreesboro, TN

The position of a particle on the x-axis at time t, with $t \geq 0$, is given by $x = f(t)$, where $f(t) = t^{(t^t)}$ if $t > 0$ and $f(0) = 0$.

(a) Let $v(t)$ be the velocity of the particle at time t. Find $\lim_{t\to0+} v(t)$.

(b) Is there more than one positive real number b such that the average acceleration of the particle over the time interval $[0,b]$ is zero?

(c) Use your answer to part (b) to determine whether there is a time $t = c$ such that the acceleration at time c is equal to the average acceleration over the time interval $[0,c]$.

Solution:

(a) Note that for $t > 0$, we have $v(t) = \left(t^{t^t}\right)' = t^{t^t}(t^t \ln t(1+\ln t) + t^{t-1})$.

$\lim_{t\to0+} v(t) = \lim_{t\to0+} \frac{f(t)-f(0)}{t} = \lim_{t\to0+} \frac{t^{(t^t)}-0}{t} = \lim_{t\to0+} t^{(t^t-1)} = \lim_{t\to0+} \exp(\ln t(t^t-1))$

Now since $\lim_{t\to0+} \exp(\ln t(t^t-1)) = \exp(\lim_{t\to0+}(\ln t(t^t-1)))$, consider $\lim_{t\to0+}(\ln t(t^t-1))$

For $t > 0$,

$\ln t(t^t-1) = \ln t(\exp(t\ln t)-1) = t\ln^2 t + (t\ln t)^3/2! + \cdots = t\ln^2 t(1+(t\ln t)/2! + \cdots$

Observe that $\lim_{t\to0+} t\ln t = 0$ (apply L'Hôpital's Rule to $\ln t/(1/t)$), hence for suitable $c > 0$, $t \in (0,c)$ implies

$|t\ln t| < 1$, then $|t\ln^2 t(1+(t\ln t)/2! + \cdots| < t\ln^2 t(1+1/2!+1/3!+\cdots) = (e-1)t\ln^2 t$.

Applying L'Hôpital's Rule to $t\ln^2 t$ we get

$\lim_{t\to0+} t\ln^2 t = \lim_{t\to0+}\left(\frac{\ln^2 t}{1/t}\right) = \lim_{t\to0+}(-2t\ln t) = 0$.

Thus $\lim_{t\to0+}(\ln t(t^t-1)) = 0$, and accordingly $\lim_{t\to0+} \exp(\ln t(t^t-1)) = \lim_{t\to0+} v(t) = 1$.

(b) Note that for $t > 0$,

$v'(t) = \left(t^{t^t}(t^t\ln t(1+\ln t)+t^{t-1})\right)^2 + t^{t^t}(t^t\ln t(1+\ln t)^2 + t^{t-1}\ln t + 2t^{t-1}(1+\ln t) - t^{t-2})$,

and this is continuous on $(0,b]$. Denote the (weighted) average acceleration over interval $[0,b]$ by $\hat{a}(b)$, then for $\epsilon > 0$,

$\hat{a}(b) = \frac{1}{b}\lim_{\epsilon\to0}\int_\epsilon^b v'(t)dt = \frac{1}{b}(v(b) - \lim_{\epsilon\to0} v(\epsilon)) = \frac{1}{b}(v(b)-1)$, so $\hat{a}(b) = 0$ whenever $v(b) = 1$. We may confirm by calculation that $v(0.1) > 1.6$, $v(0.5) < 0.8$, and $v(1.5) > 4.7$. Then by continuity of $v(t)$, there is $b_1 \in (0.1, 0.8)$ and $b_2 \in (0.8, 1.5)$ such that the average accelerations on $[0,b_1]$ and $[0,b_2]$ are zero.

(c) With b_1 and b_2 as above, Rolle's Theorem applies to $\hat{a}(b)$ on $[b_1,b_2]$, and there is some $c \in (b_1,b_2)$ such that $(\hat{a}(b))'_{b=c} = 0$. But $(\hat{a}(b))' = -\frac{1}{b^2}(v(b)-1) + \frac{1}{b}v'(b)$, so $-\frac{1}{c^2}(v(c)-1) + \frac{1}{c}v'(c) = 0$ implies $\frac{1}{c}(v(c)-1) = \hat{a}(c) = v'(c)$, i.e. the average acceleration and instantaneous acceleration at $t = c$ are equal, as required.

FGCU Problem Group
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We conclude that

$$3\,\mathrm{tr}\,(A^*A) + \mathrm{tr}\,(AAA^*A^*) = 2\,\mathrm{Re}\left[\mathrm{tr}\,(AA + AA^*A)\right]$$

if and only if $\mathrm{tr}(S^*S) + \mathrm{tr}(T^*T) = 0$. Since $\mathrm{tr}(S^*S)$ and $\mathrm{tr}(T^*T)$ are non-negative reals, $\mathrm{tr}(S^*S) + \mathrm{tr}(T^*T) = 0$ if and only if both S and T are 0. But $S = 0$ implies that A is Hermitian and $T = 0$ implies that A is idempotent.

Also solved by CON AMORE PROBLEM GROUP, The Danish University of Education, Copenhagen, Denmark; DANIELE DONINI, Bertinoro, Italy; BILL DUNN, III, Montgomery C.; OSSAMA A. SALEH and RONALD L. SMITH (jointly), U. of Tennessee-Chattanooga; NIGEL SALTS, Marymount C.; LI ZHOU, Polk C. C.; and the proposer.

The motion of an exponential particle

707. *Proposed by Dennis Walsh, Middle Tennessee State University, Murfreesboro, TN*

The position of a particle on the x-axis at time t, with $t \geq 0$, is given by $x = f(t)$ where $f(t) = t^{(t^t)}$ if $t > 0$ and $f(0) = 0$.

(a) Let $v(t)$ be the velocity of the particle at time t. Find $\lim_{t\to 0^+} v(t)$.

(b) Is there more than one positive real number b such that the average acceleration of the particle over the time interval $[0, b]$ is zero?

(c) Use your answer to part (b) to determine whether there is a time $t = c$ such that the acceleration at time c is equal to the average acceleration over the time interval $[0, c]$.

Solution by Stephen Kaczkowski, Orange County Community College, Middletown, NY

(a) Using $\exp(x)$ for e^x, we have $f(t) = \exp[\ln(t)\cdot\exp(t\ln(t))]$. Differentiating yields

$$v(t) = f'(t) = \exp\left[\ln(t)\left(t^t + t - 1\right)\right]\left[1 + t\ln(t)\left(1 + \ln(t)\right)\right].$$

Since $t[\ln(t)]^n$ has right-hand limit 0 at 0 for $n = 1$ and 2 and $\frac{(t^t+t-1)}{(\ln(t))^{-1}}$ has right-hand limit 0 at 0, we see that $\lim_{t\to 0^+} v(t) = \exp(0)[1 + 0 + 0] = 1$. Note that $v(1) = 1$.

(b) Since

$$v\left(\frac{1}{2}\right) = 2^{\left(\frac{1-\sqrt{2}}{2}\right)}\left[1 + \ln\left(\sqrt{2}\right)\ln\left(\frac{2}{e}\right)\right] < 1$$

and

$$v\left(\frac{1}{4}\right) = 2^{\left(\frac{3-2\sqrt{2}}{2}\right)}\left[1 + \ln\left(\sqrt{2}\right)\ln\left(\frac{4}{e}\right)\right] > 1,$$

there exists some value b between $1/4$ and $1/2$ such that $v(b) = 1$.

The average acceleration on $[0, t]$ is given by

$$\overline{a}(t) = \frac{\int_0^t a(x)\,dx}{t}.$$

It follows that

$$\overline{a}(t) = \frac{1}{t}\left[v(t) - \lim_{t\to 0^+} v(t)\right] = \frac{v(t)-1}{t},$$

and the average acceleration is 0 for $t = b$ and 1.

(c) Apply Rolle's Theorem to the function $\overline{a}(t)$ on the interval $[b, 1]$. Then

$$\frac{d\overline{a}(t)}{dt} = \frac{ta(t) - \int_0^t a(x)\,dx}{t^2} = \frac{a(t) - \frac{1}{t}\int_0^t a(x)\,dx}{t} = \frac{a(t) - \overline{a}(t)}{t}$$

vanishes for some c in the interval $[b, 1]$, forcing $a(c) = \overline{a}(c)$.

Also solved by HERB BAILEY and DAVID L. FINN (jointly), Rose-Hulman Institute of Technology; PHIL CLARKE, L.A. Valley C. (who solved an equivalent problem instead); DANIELE DONINI, Bertinoro, Italy; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; OVIDIU FURDUI, Western Michigan U.; WILLIAM SEAMAN, Albright C.; ALEXEY VOROBYOV, Irvine, CA; LI ZHOU, Polk C. C.; and the proposer.

A Diophantine equation with twos revisited

708. *Proposed by Alexander Faber (undergraduate), Pennsylvania State University, State College, PA*

Find all positive integers n such that

$$2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2001} + 2^n$$

is a perfect square. [This problem was inspired by problem 673 in the March 2000 issue of this journal.]

Solution by Arthur H. Foss, Plantation, FL

There is no such n. To see why, let $R = 2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2001} + 2^n = 2^{1994}241 + 2^n$, and consider the following cases.

Case 1: (n is even) Then, since 241, 2^{1994} and 2^n are congruent to $1 \pmod 3$, $R \equiv 2 \pmod 3$. Since 2 is a quadratic non-residue modulo 3, R is not a perfect square.

Case 2: (n odd and $n \le 1995$) $R = 2^{n-1}[241 \cdot 2^{1995-n} + 2]$. If $n = 1995$, then $R = 2^{1994}3^5$ which is not a perfect square. If $n < 1995$, then $2^{1995-n} \equiv 0 \pmod 4$ so that $241 \cdot 2^{1995-n} + 2 \equiv 2 \pmod 4$. Since 2 is a quadratic non-residue modulo 4 and 2^{n-1} is a perfect square, R is not a perfect square.

Case 3: (n odd and $n > 1995$) $R = 2^{1994}[241 + 2^{n-1994}]$. Now R will be a perfect square if and only if $[241 + 2^{n-1994}]$ is a perfect square. This is impossible since (i) if $n - 1994 \equiv 1 \pmod 6$, then $241 + 2^{n-1994} \equiv 5 \pmod 7$, and 5 is a quadratic non-residue modulo 7, (ii) if $n - 1994 \equiv 3 \pmod 6$, then $241 + 2^{n-1994} \equiv 6 \pmod 9$, and 6 is a quadratic non-residue modulo 9, and (iii) if $n - 1994 \equiv 5 \pmod 6$, then $241 + 2^{n-1994} \equiv 3 \pmod 9$, but 3 is a quadratic non-residue modulo 9.

Also solved by JOHN CHRISTOPHER, California State U.-Sacramento; DANIELE DONINI, Bertinoro, Italy; NATALIO H. GUERSENZVAIG, Universidad CAECE, Argentina; ERWIN JUST, Bronx C.C.; WILLIAM SEAMAN, Albright C.; LI ZHOU, Polk C.C.; and the proposer. Two incorrect or incomplete solutions were received.