College Mathematics Journal

Problem 703

Proposed by Erwin Just and Norman Schaumberger, Bronx Community College, Bronx, NY

Let $S = \{(x,y) : x,y \in Q,\ x < y,\ x^x = y^y\}$. *Note that* $(1/4, 1/2)$ *is in S. Find an accumulation point of S or prove that S has no accumulation points.*

Solution:

S accumulates at $(\frac{1}{e}, \frac{1}{e})$. Claim: Given an open disk Δ_ε of radius $\varepsilon > 0$ centered at $(\frac{1}{e}, \frac{1}{e})$, we may find a rational point of the form $\left((\frac{n+1}{n})^{-(n+1)}, (\frac{n+1}{n})^{-n}\right) \in \Delta_\varepsilon$ for suitable $n \in \mathbb{N}$. To see this note that $\lim_{n\to\infty}[(1+\frac{1}{n})^n(\frac{n+1}{n})]^{-1} = \lim_{n\to\infty}[(1+\frac{1}{n})^n]^{-1} = \frac{1}{e}$. We are therefore guaranteed an $N_\varepsilon \in \mathbb{N}$ such that $n > N_\varepsilon \Rightarrow |(\frac{n+1}{n})^{-n} - \frac{1}{e}| < \frac{\varepsilon}{2}$ and an $M_\varepsilon \in \mathbb{N}$ such that $n > M_\varepsilon \Rightarrow \left|(\frac{n+1}{n})^{-(n+1)} - \frac{1}{e}\right| < \frac{\varepsilon}{2}$. Then for $n > \max(N_\varepsilon, M_\varepsilon)$ we have $\left((\frac{n+1}{n})^{-(n+1)}, (\frac{n+1}{n})^{-n}\right) \in \Delta_\varepsilon$, establishing the claim.

Clearly $(\frac{n+1}{n})^{-(n+1)} < (\frac{n+1}{n})^{-n}$ and it remains only to be seen that $x = (\frac{n+1}{n})^{-(n+1)}$ and $y = (\frac{n+1}{n})^{-n}$ satisfy $x^x = y^y$. This is a straightforward algebra exercise which results in the common value $x^x = y^y = \left(\frac{n}{n+1}\right)^{(\frac{n}{n+1})^{n+1}}$.

FGCU Problem Group
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$$= 5^m \left(2^{2m-3} - 2^{m-2}\right) - 5^{m-1} \left(2^{2m-4} - 2^{m-2}\right)$$
$$= 10^{m-1} \left(9 \cdot 2^{m-3} - 2\right)$$

and the sum of the digits in a single cycle $\{a_{m+k,k+1}\}_{k=0}^{l-1}$ of the sequence $\{a_{m+k,k+1}\}_{k=0}^{\infty}$ is

$$\sum_{k=0}^{l-1} a_{m+k,k+1} \equiv \sum_{k=0}^{l-1} a_{m+k,k+1} 10^{m-1} = 10^{m-1} \left(9 \cdot 2^{m-3} - 2\right) \equiv -2 \equiv 7 (\text{mod } 9).$$

Also solved by DUANE M. BROLINE, Eastern Illinois U.; JOHN GRAHAM, Penn State-Wilkes-Barre; HARRIS KWONG, SUNY C. at Fredonia; MOHAMMAD RIAZI-KERMANI, Fort Hays U.; WILLIAM SEAMAN, Albright C.; LI ZHOU, Polk C. C. (Part (a)); and the proposer.

A single accumulation point

703. *Proposed by Erwin Just (emeritus) and Norman Schaumberger (emeritus), Bronx Community College, Bronx, NY*

Let $S = \{(x, y) : x, y \text{ rational}, x < y, x^x = y^y\}$. Note that $(\frac{1}{4}, \frac{1}{2})$ is in S. Find an accumulation point of S or prove that S has no accumulation points.

Solution by Ron Rietz, Gustavus Adolphus College, St Peter, MN

We will show that $(\frac{1}{e}, \frac{1}{e})$ is an accumulation point of S. Let $(x, y) \in S$ and let $r = \frac{x}{y}$. Then r is rational, $0 < r < 1$, and $x \ln(x) = y \ln(y)$. Hence $\frac{\ln(y)}{\ln(x)} = \frac{x}{y} = r$ and $y = x^r$. Thus, $x = r^{\frac{1}{(1-r)}}$ and $y = r^{\frac{r}{(1-r)}}$. Now if r is chosen so that $\frac{1}{1-r}$ and $\frac{r}{1-r}$ are positive integers, then $(x, y) = (r^{\frac{1}{1-r}}, r^{\frac{r}{1-r}}) \in S$. If $r = \frac{n}{n+1}$, n a positive integer, then $\frac{1}{1-r} = n + 1$ and $\frac{r}{1-r} = n$. Hence $(x, y) = ((\frac{n}{n+1})^{n+1}, (\frac{n}{n+1})^n)$. Since $\lim_{n\to\infty}(\frac{n}{n+1})^{n+1} = \lim_{n\to\infty}(\frac{n}{n+1})^n = \frac{1}{e}$, $(\frac{1}{e}, \frac{1}{e})$ is an accumulation point of S.

Also solved by MICHAEL ANDREOLI, Miami-Dade C. C.; MICHEL BATAILLE, Rouen, France; BRIAN BRADIE, Christopher Newport U.; DUANE M. BROLINE, Eastern Illinois U.; PHIL CLARKE, L.A. Valley C.; DANIELE DONINI, Bertinoro, Italy; FLORIDA GULF COAST UNIVERSITY PROBLEM GROUP; JOHN GRAHAM, Penn State-Wilkes-Barre; PAUL M. HARMS, North Newton, KS; RICKY IKEDA, Leeward C. C.; DAVE OHLSEN, Santa Rosa Junior C.; WILLIAM SEAMAN, Albright C.; and the proposer.

Editors' Note: Solvers Broline and Ohlsen showed that S has no other accumulation points.

Symmetric random variables

704. *Proposed by Roger B. Nelsen, Lewis & Clark College, Portland, OR*

We say that a continuous random variable is *symmetric about zero* if the density function of the random variable is an even function. Let X and Y be identically distributed continuous random variables. Prove or disprove:

(a) The difference $X - Y$ is symmetric about zero.

(b) If X and Y are symmetric about zero, then so is the sum $X + Y$.

Do your answers in (a) or (b) change, if X and Y are also assumed to be independent?

