College Mathematics Journal

Problem 695

Proposed by Andrew Cusumano, Great Neck, NY

For n a positive integer greater than 2*, define* H_n *by*
$H_n = \frac{1}{n} + \frac{1}{n+3} + \frac{1}{n+6} + \cdots + \frac{1}{n+3(n-1)}$. *Prove that the sequence* H_n *is convergent and find its limit.*

Solution:

We will construct two companion sequences which respectively majorize and minorize the given sequence and which will both be shown to converge to $\frac{2\ln 2}{3}$.

Define the majorizing sequence as follows:

For $n = 3k, 3k+1$, or $3k+2$, set
$U_n = \frac{1}{3k} + \frac{1}{3k+3} + \cdots + \frac{1}{3k+3(3k-1)} + \frac{1}{3k+3(3k)} + \frac{1}{3k+3(3k+1)} = \frac{1}{3k} + \frac{1}{3k+3} + \cdots + \frac{1}{12k-3} + \frac{1}{12k} + \frac{1}{12k+3}$

Clearly $U_n > H_n$, since U_n has at least as many summands as H_n, each of which dominates term-by-term, starting with the initial summands in each expression. We recognize that $U_n = \frac{1}{3}(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{4k+1})$ and it follows that
$\int_k^{4k+2} \frac{dx}{x} > \sum_{j=k}^{4k+1} j^{-1} = 3U_n$. Hence $\ln(\frac{4k+2}{k}) > 3U_n$, and
$\frac{1}{3}\lim_{k\to\infty}(\ln(\frac{4k+2}{k})) \geq \lim_{n\to\infty} U_n \geq \lim_{n\to\infty} H_n$. Taking limits, we obtain $\frac{2\ln 2}{3} \geq \lim_{n\to\infty} H_n$.

Now define the minorizing sequence as follows:

For $n = 3k, 3k-1$, or $3k-2$, set
$L_n = \frac{1}{3k} + \frac{1}{3k+3} + \cdots + \frac{1}{3k+3(3k-1)-6} = \frac{1}{3k} + \frac{1}{3k+3} + \cdots + \frac{1}{12k-9}$

Clearly $L_n < H_n$, since H_n has at least as many summands as L_n, each of which dominates term-by-term, starting with the initial summands in each expression, exactly as above. We recognize again that $L_n = \frac{1}{3}(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{4k-3})$. It follows that
$\int_k^{4k-2} \frac{dx}{x} < \sum_{j=k}^{4k-3} j^{-1} = 3L_n$. Hence $\ln(\frac{4k-2}{k}) < 3L_n$, and
$\frac{1}{3}\lim_{k\to\infty}(\ln(\frac{4k-2}{k})) \leq \lim_{n\to\infty} L_n \leq \lim_{n\to\infty} H_n$. Again taking limits we get $\frac{2\ln 2}{3} \leq \lim_{n\to\infty} H_n$.

Combining results it is apparent that $\lim_{n\to\infty} H_n = \frac{2\ln 2}{3}$.

Remark:

We note that the sequence is not monotone decreasing, for example $H_{848} > H_{847}$. Moreover, the restriction on n seems unnecessary (as long as $n \in \mathbb{N}$) since the limit behavior does not depend on initial values, provided they are defined. In this case they are, and the usual convention applies...if $n = 1$ then $n + 3(n-1) = n$ and the sum has one term.

This result generalizes. Define for $n, m \in \mathbb{N}$, $H_{n,m} = \frac{1}{n} + \frac{1}{n+m} + \frac{1}{n+2m} + \cdots + \frac{1}{n+m(n-1)}$.
Then it can be shown by the method above that for fixed m, $\lim_{n\to\infty} H_{n,m} = \frac{\ln(m+1)}{m}$.

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form $n = jp^i - 1$. Let p^i be the highest power of p that is less than or equal to n. Since $n \geq p$, we know that $i \geq 1$. Then $n = jp^i + k$, with $0 < j < p$ and $0 \leq k < p^i - 1$. Now, $\binom{n}{k+1} = \binom{n}{k}\frac{jp^i}{k+1}$. Since $k+1 < p^i$, $\frac{jp^i}{k+1}$ has a factor of p in its numerator that is not cancelled by a factor in its denominator. Therefore, $\binom{n}{k+1}$ is divisible by p.

(c) For each positive exponent $i < m$, there are $p-1$ such n of the form $jp^i - 1$ with $1 \leq j < p$. There are $p-2$ such n of the form $jp^0 - 1$, and there is the single value $n = p^m - 1$. This gives a total of $m(p-1)$ such values of n.

Also solved by MARC BRODIE, C. of St. Benedict; BRUCE COLLINGS, Brigham Young U.; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; TOMMY GOEBELER, Briarwood, NY; JOHN GRAHAM, Penn State Wilkes-Barre; CARL HURD, Altoona C. of Penn State U.; RICKY IKEDA, Leeward C. C.; STEPHEN KACZKOWSKI, Orange County C.C.; HARRIS KWONG, SUNY C. at Fredonia; PHILIP OPPENHEIMER, Norwalk, CT; ROBERT PATENAUDE, C. of Canyons; WILLIAM SEAMAN, Albright C.; ALEXEY VOROBYOV, Irvine, CA; LI ZHOU, Polk C.C.; and the proposer. Solutions to parts (a) and (b) were submitted by BRIAN BRADIE, Christopher Newport U.; ANDREW IANNACCONE (student), Harvey Mudd C.; ERIC MALM (student), Saint George's School, Spokane, WA; and JOHN S. SUMNER and KEVIN L. DOVE (jointly), U. of Tampa; and the proposer.

Editors' Note: The proposer and several solvers gave answers to part (b) in terms of the base p presentation of the integer n. In this context, the answer to part (b) is that n must have base p presentation $(n_t n_{t-1} \ldots n_1 n_0)_p$ with $n_k = p-1$ for $k < t$. Donini and Kwong provided several references that are relevant to this approach; among these is N. J. Fine, Binomial coefficients modulo a prime, *American Mathematical Monthly* **54** (1947), 589–592.

Sequential convergence

695. *Proposed by Andrew Cusumano, Great Neck, NY*

For n a positive integer greater than 2, define H_n by

$$H_n = \frac{1}{n} + \frac{1}{n+3} + \frac{1}{n+6} + \ldots + \frac{1}{n+3(n-1)}.$$

Prove that the sequence H_n is convergent and determine its limit.

Solution by Charles Diminnie and Karl Havlak, Angelo State University, San Angelo, TX

More generally, consider the sequence defined by

$$H_n(a) = \frac{1}{n} + \frac{1}{n+a} + \frac{1}{n+2a} + \ldots + \frac{1}{n+a(n-1)}$$

for n any integer greater than 2 and a any positive real number. Note

$$H_n(a) = \frac{1}{n}\left(1 + \frac{1}{1+\frac{a}{n}} + \frac{1}{1+\frac{2a}{n}} + \cdots + \frac{1}{1+\frac{a(n-1)}{n}}\right)$$

which is the left Riemann sum of $f(x) = \frac{1}{1+ax}$ with partition $\left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ of $[0, 1]$. Thus,

$$\lim_{n\to\infty} H_n(a) = \int_0^1 \frac{1}{1+ax}dx = \frac{1}{a}\ln(1+a)$$

Also solved by REZA AKHLAGHI, Prestonsburg C.C.; SIHAM ALFRED, Raritan Valley C.C.; MICHAEL ANDREOLI, Miami-Dade C.C.; HERB BAILEY, Rose-Hulman Institute of Technology; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; TOM BEATTY'S Calculus II class, Florida Gulf Coast U.; THE BOSTON COLLEGE PROBLEM GROUP, Boston C.; PAUL BRACKEN, U. de Montreal; BRIAN BRADIE, Christopher Newport U.; MARC BRODIE, C. of St. Benedict; STAN BYRD and CHRISTOPHER P. MAWATA (jointly), U. of Tennessee at Chattanooga; HONGWEI CHEN, Christopher Newport U.; JOHN CHRISTOPHER, California State U.; MARGARET CIBES, Trinity C.; PHIL CLARKE, L.A. Valley C.; CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; JIM DELANY, California Polytechnic State U.; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; BILL DUNN III, Montgomery C.; RUSS EULER and JAWAD SADEK (jointly), Northwest Missouri State U.; NORA FRANZOVA, Harford C.C.; GEORGE WASHINGTON UNIVERSITY PROBLEMS GROUP, George Washington U.; TOMMY GOEBELER and JESSICA CHONG (jointly), Briarwood, NY; JOHN GRAHAM, Penn State Wilkes-Barre; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; PAUL M. HARMS, North Newton, KS; JIM HEWITT, Inver Hills C.C.; JAMES HUGHES (student), U. of Tennessee; CARL HURD, Altoona C. of Penn State U.; DOUGLAS IANNUCCI, U. of Virgin Islands; RICKY IKEDA, Leeward C. C.; PETER M. JARVIS, Georgia C. & State U.; STEPHEN KACZKOWSKI, Orange County C.C.; A. BATHI KASTURIARACHI, Kent State U.-Stark; MURRAY S. KLAMKIN, U. of Alberta; HARRIS KWONG, SUNY C. at Fredonia; JASON LEE (student), U. of California-San Diego; JEROLD LEWANDOWSKI, Rensselaer Polytechnic Institute; DENNIS LUCEY, Western Maryland C.; ERIC MALM (student), Saint George's School, Spokane, WA; JUAN-BOSCO ROMERO MARQUEZ, Universidad de Valladolid, Valladolid, Spain; KEVIN MCDOUGAL, U. of Wisconsin-Oshkosh; MOHAMUD MOHAMMED Temple U.; KANDASAMY MUTHUVEL, U. of Wisconsin-Oshkosh; GREG NEUMER, Oak Park, IL; PHILIP OPPENHEIMER, Norwalk, CT; RON PERSKY, Christopher Newport U.; ARTHUR J. ROSENTHAL, Salem State C.; R. P. SEALY, Mount Allison U.; WILLIAM SEAMAN, Albright C.; JOHN S. SUMNER and KEVIN L. DOVE (jointly), U. of Tampa; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; LYNDSEY VAN WORMER (student), Alma C.; THOMAS VANDEN EYNDEN, Thomas More C.; M. VOWE, Therwil, Switzerland; THOMAS C. WALES, Cambridge, MA; TRACY WANG, Raritan Valley C.C.; ROB WILLIAMS, Alfred U.; YAJUN YANG, SUNY-Farmingdale; LI ZHOU, Polk C.C.; and the proposer.

Editors's Note: SAAD ANAN of Elizabeth City, NC should have been included in the list of solvers of Problem 684 which appeared in the September 2001 issue. He noted, in his solution, that the hypothesis $z = |PQ''R'| = |QR''P'| = |RP''Q'|$ is redundant and also provided an extension of the result.

"Help! Solve Me!" the Equation Cried

Ralph Raimi (rarm@math.rochester.edu) noticed the following on page 279 of *The Educated Child*, by William Bennett, Chester E. Finn, Jr., and John T. E. Cribb, Jr. (Free Press, 1999):

> The expression $2x + 6 = 16$ is a neat and concise way of saying, "Find the value of x such that when x is multiplied by two and six is added to that product, the resulting sum is 16.

Professor Raimi comments, "Of course, the expression says no such thing."

There seems to be abroad in the land the impression that symbols like "+" and "=" are imperatives, commanding us to *do* something. It is curious.