College Mathematics Journal

Problem 678 - May 2000

Proposed by David Atkinson, Olivet Nazarene University, Kankakee, IL

For $n = 0, 1\ldots,$ *find the value of the double sum* $\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{(-1)^j}{i!j!}$ *as a function of* n.

Solution:

Let $S(n)$ denote the sum, and $a(i,j)$ be the general term. We show that $S(n) = 1$, for all $n \in \mathbb{N}$. We proceed by induction. $S(0) = 1$, by direct calculation. Assume $S(n-1) = 1$. The terms of the sum may be viewed as elements in an infinite array, only a portion of which are considered depending on the value of n. The sum $S(n-1)$ adds all terms in row i from $j = 0$ out to $j = n-1$. Clearly the last row counted has index $(n-1)$, and there is but one term taken from it into the sum, namely $a(n-1,0)$.

Now consider $S(n)$. This sum extends each row taken into $S(n-1)$ by one entry, namely that of the form $a(i,n-i)$. Thus we have...

$S(n) = S(n-1) + a(0,n) + a(1,n-1) + \cdots + a(k,n-k) + \cdots a(n-1,1) + a(n,0) =$

$S(n-1) + \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}$.

But by the Binomial Theorem, $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} = (1 + (-1))^n = 0$, hence $\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} = 0$, and finally $S(n) = 1$. This establishes the induction argument, and we conclude $\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{(-1)^j}{i!j!} = 1$, independently of n.

J. OSLER, Rowan U.; WILLIAM SEAMAN, Albright C.; R. S. TIBERIO, Natick, MA; SAMUEL A. TRUITT, Jr., Middle Tennessee State U.; THOMAS VANDEN EYNDEN, Thomas More C.; DOUG WILCOCK, Cape Cod Academy, MA; LI ZHOU, Polk C. C.; and the proposer.

An Inverse Function

677. *Proposed by Geoffrey A. Kandall. Hamden, CT*

The function $f : (0, \infty) \to (-\infty, \infty)$ defined by $f(t) = \frac{\sinh(2t)}{2\sinh(t)} - \coth(t)$ is increasing and onto. Derive an explicit formula, that involves only algebraic functions and natural logarithms, for the inverse function f^{-1}.

Solution by M. Reza Akhlaghi, Prestonsburg Community College, Prestonsburg, KY

The function f satisfies

$$y = f(t) = \frac{(1+e^{2t})(e^{2t} - 2e^t - 1)}{2e^t(e^{2t} - 1)}$$

with $f(\ln(1+\sqrt{2})) = 0$. Let $u = e^t$. Solving for u in terms of y, we are led to

$$u^4 - 2(y+1)u^3 + 2(y-1)u - 1 = 0.$$

This equation factors:

$$\left(u^2 - (y+1)u - y - \sqrt{y^2+1}(u+1)\right)\left(u^2 - (y+1)u - y + \sqrt{y^2+1}(u+1)\right) = 0.$$

The fact that $y = 0$ when $u = 1+\sqrt{2}$ shows that only the left factor will yield a solution; using the quadratic formula it also shows that

$$u = \frac{1}{2}\left(y + 1 + \sqrt{y^2+1} + \sqrt{\left(y+1+\sqrt{y^2+1}\right)^2 + 4\left(y + \sqrt{y^2+1}\right)}\right)$$

is the only acceptable solution. The desired function is $t = f^{-1}(y) = \ln(u)$.

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; JOSEPH COSTER, Macomb, IL; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U; JOHN GRAHAM, Penn State Wilkes-Barre; MURRAY S. KLAMKIN, U. of Alberta; HARRIS KWONG, SUNY C. at Fredonia; KIM McINTURFF, Santa Barbara, CA; STEPHEN NOLTIE, Ohio U.- Lancaster; WILLIAM SEAMAN, Albright C.; CORNELIUS STALLMAN and GERALD THOMPSON, Augusta State U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; OMER YAYENIE and MOHAMUD MOHAMMED, Temple U.; LI ZHOU, Polk C.C.; and the proposer.

A Double Sum

678. *Proposed by David Atkinson, Olivet Nazarene University, Kankakee, IL*

For $n = 0, 1, \ldots$, find the value of the double sum $\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{(-1)^j}{i!j!}$ as a function of n.

Solution by Dennis Walsh, Middle Tennessee State University, Murfreesboro, TN

The double sum, say $D(n)$, is simply a fancy way of expressing the number 1. To see this, we introduce a change of variables. Let $r = i$ and $s = i + j$. Then $j = s - r$, giving $s = r$ when $j = 0$ and $s = n$ when $j = n - i$. Therefore,

$$D(n) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{(-1)^j}{i!j!} = \sum_{r=0}^{n}\sum_{s=r}^{n} \frac{(-1)^{s-r}}{r!(s-r)!}.$$

Now switch the order of summation so that

$$D(n) = \sum_{s=0}^{n}\sum_{r=0}^{s} \frac{(-1)^{s-r}}{r!(s-r)!},$$

and then multiply the inner sum by $\frac{s!}{s!}$ to get

$$D(n) = \sum_{s=0}^{n} \frac{1}{s!} \sum_{r=0}^{s} \binom{s}{r} (1)^r (-1)^{s-r}.$$

When $s > 0$, the inner sum is the binomial expansion of $(1-1)^s$, which is identically 0, and when $s = 0$, the inner sum reduces to $\binom{0}{0}1^0(-1)^0$, or 1. Hence $D(n) = \frac{1}{0!}(1) + \sum_{s=1}^{n} \frac{1}{s!}(0) = 1 + 0 = 1$.

Also solved by M. REZA AKHLAGHI, Prestonsburg C. C.; TEWODROS AMDEBERHAN, DeVry Institute of Technology; MICHAEL H. ANDREOLI, Miami-Dade C. C.; DAVID AUKERMAN (student), Taylor U.; RAJESH K. BARNWAL, Middle Tennessee State U.; MICHEL BATAILLE, Rouen, France; RICH BAUER, Shoreline, WA; BRIAN D. BEASLEY, Presbyterian C.; JEREMY CASE, Taylor U.; JOSEPH COSTER, Macomb, IL; CHARLES R. DIMINNIE, Angelo State U.; DANIELE DONINI, Bertinoro, Italy; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, III, Montgomery C.; MORDECHAI FALKOWITZ, Hamilton, Ontario, Canada; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U; JOHN GRAHAM, Penn State Wilkes-Barre; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; PAUL M. HARMS; PETER M. JARVIS, Georgia C. & State U.; LEE KHING KELLER, JOEL PEARSON, and JOANNA RISING (students) West Chester U.; HARRIS KWONG, SUNY C. at Fredonia; KEE-WAI LAU, Hong Kong, China; KIM McINTURFF, Santa Barbara, CA; DARRELL P. MINOR, Columbus State C. C.; MOHAMUD MOHAMMED and AKALU TEFERA, Temple U.; STEPHEN NOLTIE, Ohio U.- Lancaster; PHILLIP OPPENHEIMER, South Norwalk, CT; THOMAS J. OSLER, Rowan U.; C.I. PETROS, St. Lambert, Quebec, Canada; ROB PRATT, U. of North Carolina at Chapel Hill; YANIR A. RUBINSTEIN, Haifa, Israel; IAN D. RUTHERFORD, Mount Allison U.; WILLIAM SEAMAN, Albright C.; JOHN HENRY STEELMAN, Indiana U. of Pennsylvania; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; MICHAEL VOWE, Therwil, Switzerland; THOMAS C. WALES, Cambridge, MA; JACK V. WALES, JR., The Thacher School, Ojai, CA; HANSRUEDI WIDMER, Nussbaumen, Switzerland; LI ZHOU, Polk C. C.; and the proposer.

Editors' Note: In addition to the solution used above, Walsh provided three alternate solutions based on probability, generating functions, and induction.

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