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Proposed by Geoffrey A. Kandall, Hamden, CT

The function $f : (0,\infty) \to (-\infty,\infty)$ *defined by* $f(t) = \frac{\sinh(2t)}{2\sinh(t)} - \coth(t)$ *is increasing and onto. Derive an explicit formula, that involves only algebraic functions and natural logarithms, for the inverse function* f^{-1}.

Solution:

$f(t) = \frac{2\sinh(t)\cosh(t)}{2\sinh(t)} - \frac{\cosh(t)}{\sinh(t)} = \frac{e^t+e^{-t}}{2} - \frac{e^t+e^{-t}}{e^t-e^{-t}}$. We may assume that $y = f^{-1}(t)$ exists, and it follows that $\frac{e^y+e^{-y}}{2} - \frac{e^y+e^{-y}}{e^y-e^{-y}} = t$. Let $z = e^y$, then substituting into the previous expression, we have $\frac{1}{2}(z+z^{-1}) - \frac{z+z^{-1}}{z-z^{-1}} = t$, and after clearing fractions $z^2 - z^{-2} - 2(z+z^{-1}) = 2t(z-z^{-1})$. Multiplying thru by z^2 and collecting terms we finally have $z^4 - 2(t+1)z^3 + 2(t-1)z - 1 = 0$. This quartic factors as follows:

$$\left(z^2 - (1+t+\sqrt{1+t^2}\,)z - (t+\sqrt{1+t^2}\,)\right)\left(z^2 - (1+t-\sqrt{1+t^2}\,)z - (t-\sqrt{1+t^2}\,)\right),$$

The discriminant for the second quadratic factor is $\Delta_2(t) = (1+t-\sqrt{1+t^2}\,)^2 + 4(t-\sqrt{1+t^2}\,)$, which can be negative for some values of $t \in (0,\infty)$. For instance, $\Delta_2(3) < 0$. But z must be real for all admissible t, so accordingly we reject both roots of the second factor as leading to an inversion formula.

The discriminant for the first quadratic factor is $\Delta_1(t) = (1+t+\sqrt{1+t^2}\,)^2 + 4(t+\sqrt{1+t^2}\,)$, which is easily seen to be positive for $t \in (0,\infty)$. The two real roots are then:

$$z_1 = \frac{1}{2}\left((1+t+\sqrt{1+t^2}\,) + \sqrt{(1+t+\sqrt{1+t^2}\,)^2 + 4(t+\sqrt{1+t^2}\,)}\right), \text{ and}$$

$$z_2 = \frac{1}{2}\left((1+t+\sqrt{1+t^2}\,) - \sqrt{(1+t+\sqrt{1+t^2}\,)^2 + 4(t+\sqrt{1+t^2}\,)}\right)$$

Again, we narrow the field by observing that $\sqrt{\Delta_1(t)} > 1+t+\sqrt{1+t^2}$ for all t, and hence $z_2 < 0$. This leaves $z = z_1 > 0$ as the root yielding the inversion. We have $y = \ln z$, or by the specification of the problem:

$$f^{-1}(t) = \ln\left(\frac{1}{2}\left((1+t+\sqrt{1+t^2}\,) + \sqrt{(1+t+\sqrt{1+t^2}\,)^2 + 4(t+\sqrt{1+t^2}\,)}\right)\right).$$

An Inverse Function

677. *Proposed by Geoffrey A. Kandall, Hamden, CT*

The function $f : (0, \infty) \to (-\infty, \infty)$ defined by $f(t) = \frac{\sinh(2t)}{2\sinh(t)} - \coth(t)$ is increasing and onto. Derive an explicit formula, that involves only algebraic functions and natural logarithms, for the inverse function f^{-1}.

Solution by M. Reza Akhlaghi, Prestonsburg Community College, Prestonsburg, KY

The function f satisfies

$$y = f(t) = \frac{(1 + e^{2t})(e^{2t} - 2e^{t} - 1)}{2e^{t}(e^{2t} - 1)}$$

with $f(\ln(1 + \sqrt{2})) = 0$. Let $u = e^{t}$. Solving for u in terms of y, we are led to

$$u^4 - 2(y + 1)u^3 + 2(y - 1)u - 1 = 0.$$

This equation factors:

$$\left(u^2 - (y + 1)u - y - \sqrt{y^2 + 1}(u + 1)\right)\left(u^2 - (y + 1)u - y + \sqrt{y^2 + 1}(u + 1)\right) = 0.$$

The fact that $y = 0$ when $u = 1 + \sqrt{2}$ shows that only the left factor will yield a solution; using the quadratic formula it also shows that

$$u = \frac{1}{2}\left(y + 1 + \sqrt{y^2 + 1} + \sqrt{\left(y + 1 + \sqrt{y^2 + 1}\right)^2 + 4\left(y + \sqrt{y^2 + 1}\right)}\right)$$

is the only acceptable solution. The desired function is $t = f^{-1}(y) = \ln(u)$.

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian C.; JOSEPH COSTER, Macomb, IL; DANIELE DONINI, Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U; JOHN GRAHAM, Penn State Wilkes-Barre; MURRAY S. KLAMKIN, U. of Alberta; HARRIS KWONG, SUNY C. at Fredonia; KIM McINTURFF, Santa Barbara, CA; STEPHEN NOLTIE, Ohio U.- Lancaster; WILLIAM SEAMAN, Albright C.; CORNELIUS STALLMAN and GERALD THOMPSON, Augusta State U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; OMER YAYENIE and MOHAMUD MOHAMMED, Temple U.; LI ZHOU, Polk C.C.; and the proposer.

A Double Sum

678. *Proposed by David Atkinson, Olivet Nazarene University, Kankakee, IL*

For $n = 0, 1, \ldots,$ find the value of the double sum $\sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{(-1)^j}{i!j!}$ as a function of n.