College Mathematics Journal

Problem 667 - February 2000

Solution:
We evaluate $I = \int_0^\pi \frac{2+2\cos x-\cos(n-1)x-2\cos nx-\cos(n+1)x}{1-\cos 2x}\,dx.$

Since $\cos(n-1)x = \cos nx\cos x + \sin nx\sin x$ and $\cos(n+1)x = \cos nx\cos x - \sin nx\sin x$, the integral reduces to

$$\int_0^\pi \frac{2+2\cos x-(2\cos nx\cos x+2\cos nx)}{1-\cos 2x}\,dx = \int_0^\pi \frac{2(1+\cos x)-2\cos nx(1+\cos x)}{2\sin^2 x}\,dx = \int_0^\pi \frac{(1+\cos x)(1-\cos nx)}{(1+\cos x)(1-\cos x)}\,dx = \int_0^\pi \frac{(1-\cos nx)}{(1-\cos x)}\,dx.$$

Moreover, since the integrand is an even function, $I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{(1-\cos nx)}{(1-\cos x)}\,dx$

This latter integral can be evaluated by setting $z = e^{ix}$,

then $\cos x = \frac{1}{2}(z+z^{-1})$, $\cos nx = \frac{1}{2}(z^n+z^{-n})$, and $dx = \frac{dz}{iz}$.

We have

$$\int_{-\pi}^{\pi} \frac{(1-\cos nx)}{(1-\cos x)}\,dx = \oint_{|z|=1} \frac{1-\frac{1}{2}(z^n+z^{-n})}{1-\frac{1}{2}(z+z^{-1})}\frac{dz}{iz} = \oint_{|z|=1} \frac{(z)(z^{2n}-2z^n+1)}{(z^n)(z^2-2z+1)}\frac{dz}{iz} = \frac{1}{i}\oint_{|z|=1} \frac{(z^n-1)^2}{(z^n)(z-1)^2}\,dz.$$

The Laurent series about zero for the integrand is now apparent, since

$$\frac{(z^n-1)^2}{(z^n)(z-1)^2} = \frac{1}{z^n}\left(\sum_{k=0}^{n-1} z^k\right)^2.$$

Zero is an n^{th} order pole and the coefficient of z^{n-1} in the squared sum is the corresponding residue. Observe that this coefficient is just n, hence

$\frac{1}{i}\oint_{|z|=1} \frac{(z^n-1)^2}{(z^n)(z-1)^2}\,dz = \frac{1}{i}(2\pi i)(n) = 2n\pi$. Finally, we have $I = \frac{1}{2}(2n\pi) = n\pi$.

A Trigonometric Integral

667. *Proposed by Mulatu Lemma, Savannah State University, Savannah, GA*
Find the value of the integral

$$\int_0^{\pi} \frac{2 + 2\cos(x) - \cos((n-1)x) - 2\cos(nx) - \cos((n+1)x)}{1 - \cos(2x)}\, dx$$

as a function of n, where n is a nonnegative integer.

Solution by Li Zhou, Polk Community College, Winter Haven, FL

Denote the given integral by $f(n)$. Then, it is easy to see that $f(0) = 0$ and $f(1) = \pi$. Moreover, we notice for $n > 0$,

$$\begin{aligned}
&f(n+1) - 2f(n) + f(n-1) \\
&= \int_0^{\pi} \frac{\cos((-2+n)\,x) - 2\cos(n\,x) + \cos((2+n)\,x)}{-1+\cos(2\,x)}\,dx \\
&= \int_0^{\pi} \frac{2\cos(n\,x)\left(-1+\cos(x)^2 - \sin(x)^2\right)}{-1+\cos(2\,x)}\,dx \\
&= \int_0^{\pi} 2\cos(n\,x)\,dx = 0.
\end{aligned}$$

Therefore, $f(n+1) - f(n) = f(n) - f(n-1)$, i.e. $f(n)$ is an arithmetic sequence with fixed difference π. Hence $f(n) = n\pi$ for $n \geq 0$.

Also solved by ANURAG AGARWAL, SUNY at Buffalo; ALMA COLLEGE PROBLEM SOLVING GROUP, Alma C.; TEWODROS AMDEBERHAN, DeVry Institute of Technology; MICHAEL H. ANDREOLI, Miami-Dade C. C. (North); THOMAS BASS, Carson-Newman C.; MICHEL BATAILLE, Rouen, France; MICHAEL S. BECKER and CHARLES K. COOK, U. of South Carolina at Sumter; ALPER CAY, Erciyes U., Turkey; CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; PAUL DEIERMANN, Lindenwood U.; JIM DELANY, Cal Poly San Luis Obispo; WAYNE V. DENNIS, Palos Verdes Peninsula, CA; CHARLES R. DIMINNIE, Angelo State U.; DANIELE DONINI Bertinoro, Italy; JAMES DUEMMEL, Bellingham, WA; WILLIAM DUNHAM and ELYN RYKKEN, Muhlenberg C.; BILL DUNN, Spring, TX; NEIL ECKLUND, Centre C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U.; JOHN GRAHAM, Penn State Wilkes-Barre; DAWIT HAILE, Virginia St. U.; LANCE E. HEMLOW, Raritan Valley C. C.; PETER M. JARVIS, Georgia C. & State U.; STEVE KIFOWIT, Prairie St. C.; KEE-WAI LAU, Hong Kong, China; ROBERT LAVELLE, Iona C.; JOSEPH C. LAZZARA, Fairfield, NJ; COLLEEN LIVINGSTON, Bemidji St. U.; PHILLIP OPPENHEIMER, South Norwalk, CT; WILLIAM SEAMAN, Albright C.; MICAH SMUKLER (student), Harvey Mudd C.; DANIELA SZATMARI-VOICU, U. of Calgary; AKALU TEFERA, Temple U.; SAMUEL A. TRUITT, JR., Middle Tennessee State U.; MICHAEL VOWE, Muttenz, Switzerland; HANSRUEDI WIDMER, Nussbaumen, Switzerland; JOSEPH WIENER, The U. of Texas – Pan American; and the proposer.

A Diophantine Equation with Tens

668. *Proposed by Joung-kuen Lim and Ho-joo Lee (students), Seoul National University and Kwangwoon University (respectively), Seoul, Korea*

Find all prime numbers a, b, c and d such that $10^a + b^2 = c^{10} + d^2$.

Solution by Jim Delany, California Polytechnic State University, San Luis Obispo, CA

The only example is $10^3 + 7^2 = 2^{10} + 5^2$. We begin by showing that $d \neq 2$. Assume the contrary, that $d = 2$ and $10^a + b^2 = c^{10} + 4$. If $a = 2$ then $(c^5 + b)(c^5 - b) = 96$. If so, since $c^5 \leq 96$, we must have $c = 2$, forcing $b^2 = 928$, an impossibility. Hence $a \geq 3$. Note that if either one of b or c is even, so is the other. However, $b = c = 2$