College Mathematics Journal

Problem 660 - September 1999

Proposed by Ho-joo Lee, Kwangwoon University, South Korea

Find all real numbers a such that the equation $z^3 + az + 1 = 0$ has a solution of the form $p + q\sqrt{-3}$ where p and q are positive rational numbers.

Solution:

Note that $z \neq 0$, so we may set $a = -\frac{z^3+1}{z}$, from which it follows that $\text{Im}(z^*(z^3+1)) = 0$, since $a \in \mathbb{R}$. Substituting $p + q\sqrt{-3}$ for z gives

$$z^*(z^3+1) = p^4 + 2p^3q\sqrt{-3} + 6pq^3\sqrt{-3} - 9q^4 + p - q\sqrt{-3}$$

and we see that $2p^3q + 6pq^3 - q = 0$, and since $q \neq 0$, $2p^3 + 6pq^2 - 1 = 0$. The only positive rational roots this cubic in p can have are 1 and $\frac{1}{2}$. If $p = 1$, $q = \frac{\sqrt{2}}{2} \notin \mathbb{Q}$, and if $p = \frac{1}{2}$, $q = \frac{1}{2}$.

Since $a = \frac{-z^*(z^3+1)}{z^*z} = \frac{-(p^4-9q^4+p)}{p^2+3q^2}$, substituting $p = q = \frac{1}{2}$, we find the only possible value for a is zero.

is established using arguments much like those in the following solution. In the list of solvers given below, we identify solvers who replaced the given condition (a) or the given condition (b) with a condition that is equivalent to the remaining condition.

Solution by Kee-Wai Lau, Hong Kong, China

We may assume that $m \geq 2$ and that the t_i are distinct. Let $f(x) = (\sum_{k=1}^{m} a_k x^{t_k}) - 1$ so that $f(0) = -1 < 0$, $f(x) > 0$ for large x and $f'(x) = (\sum_{k=1}^{m} a_k t_k x^{t_k - 1}) > 0$ for $x > 0$. Thus, the given equation has a unique positive solution. Let z be a complex solution and suppose that $|z| < w$. Then we obtain the contradiction

$$1 = \left| \sum_{k=1}^{m} a_k z^{t_k} \right| \leq \sum_{k=1}^{m} a_k |z^{t_k}| < \sum_{k=1}^{m} a_k w^{t_k} = 1.$$

To see that (a) and (b) are not equivalent, consider $1 = 4x^4 + 3x^2$ which has roots $\pm \frac{1}{2}$ and $\pm i$. Then (a) is true while (b) is false. However, the following condition is equivalent to the given condition (b).

(a)* If $z \neq w$ is a solution to $1 = a_1 x^{t_1} + a_2 x^{t_2} + \cdots + a_m x^{t_m}$, then $|z| > w$ where w is the unique positive real solution to the given equation.

Suppose $\gcd(t_1, t_2, \ldots, t_m) = d > 1$. Then $z = we^{2\pi i/d}$ is a root other than w of the given equation, contradicting (a)*. Now suppose that (b) holds and that $z \neq w$ is a solution to the given equation with $|z| = w$. Let $z = we^{i\theta}$ where $0 < \theta < 2\pi$. Since w is a solution to the given equation, we have $\sum_{k=1}^{m} a_k w^{t_k}(1 - e^{i\theta t_k}) = 0$. Considering the real part of this equality, we have, for $k = 1, 2, \ldots, m$, $1 - \cos(t_k \theta) = 0$ which implies that $t_k \theta = 2 n_k \pi$ where n_k is an integer that is not divisible by t_k. For $k = 1, 2, \ldots, m$, let $d_k = \gcd(t_k, n_k)$, $s_k = \frac{t_k}{d_k}$ and $r_k = \frac{n_k}{d_k}$. Then $s_k \theta = 2 r_k \pi$, $\gcd(s_k, r_k) = 1$ and $s_k > 1$. Since $\frac{\theta}{2\pi} = \frac{r_k}{s_k}$ for each k, we see that $r_j s_k = r_k s_j$ for $k, j = 1, 2, \ldots, m$. It follows that $s_j | s_k$ and $s_k | s_j$ so that there is s with $s_k = s$ for $k = 1, 2, \ldots, m$, and then $\gcd(t_1, t_2, \ldots, t_m) \geq s > 1$, contradicting (b).

Also solved by MICHEL BATAILLE, Rouen, France; DANIELE DONINI, Bertinoro, Italy; and JAMES DUEMMEL, Bellingham, WA, each of whom replaced (b) with (b)*; by STEPHEN PENRICE, Morristown, NJ (who replaced (a) with (a)*; and by MATT FOSS, North Hennepin C. C.; RAYMOND N. GREENWELL, Holstra U.; GEORGE D. MATTHEWS, Indianapolis, IN; LI ZHOU, Polk C.C.; D.P. STANFORD, C. of William and Mary (who gave a solution based on the Perron-Frobenius theory of non-negative matrices); and the proposers.

A Consequence of FLT

660. *Proposed by Ho-joo Lee (student), Kwangwoon University, South Korea*

Find all real numbers a such that the equation $z^3 + az + 1 = 0$ has a solution of the form $p + q\sqrt{-3}$ where p and q are positive rational numbers.

Solution by Blair K. Spearman, Okanagan University College, Kelowna, B.C., Canada

Since the coefficients of the given cubic polynomial are real, and the coefficient of z^2 equals zero, it follows that the three roots of the given equation are $r_1 = p + q\sqrt{-3}$, $r_2 = p - q\sqrt{-3}$ and $r_3 = -2p$. Examining the constant term in the equation we note that

$$1 = -r_1 r_2 r_3 = 2p(p^2 + 3q^2) = (p+q)^3 + (p-q)^3.$$

Therefore $(x, y) = (p+q, p-q)$ is a rational point on $x^3 + y^3 = 1$, the Fermat curve of exponent 3. Since the only rational points on this curve are $(x, y) = (1, 0)$ or $(0, 1)$, and p and q are positive, we determine that $p = q = \frac{1}{2}$. Therefore $r_1 = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$, satisfying the requirements of the problem, and $r_3 = -1$. Substituting $r_3 = -1$ into the given equation we find that the only real number a satisfying the required conditions is $a = 0$.

Also solved by REZA AKHLAGHI, Prestonsburg C.C. and FARY SAMI, Harford C.C.; BEN B. BOWEN, Vallejo, CA; JOHN CHRISTOPHER, California State U., Sacramento; CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; JOHN COOPER, RENU CHANDRA, LAUREN BOOB (students), Shippensburg U.; DANIELE DONINI, Bertinoro, Italy; BILL DUNN, III, Montgomery C.; FLORIDA GULF COAST PROBLEM GROUP, Florida Gulf Coast U.; JOHN F. GOEHL, JR., Barry U.; JOE HOWARD, New Mexico Highlands U.; RICKEY IKEDA, Leeward C.C.; ARLO W. SCHURLE, University of Guam; WILLIAM SEAMAN, Albright C.; MONTE J. ZERGER, Adams State C.; and the proposer. One incomplete and three incorrect solutions were received.

Editors' Note: William Seaman of Albright College has pointed out that the solution of Problem 631 [CMJ 30:4 (1999) 319–320] is incomplete in the case that the odd integer k in the published solution has the property that k^2 can be represented in more than one way as a sum of two squares. There are ten odd integers k less than 261 such that k can be written as a sum of squares and k^2 has multiple representations as a sum of two squares; these are 25, 65, 85, 125, 145, 169, 185, 205, 221 and 225. It is straightforward to check that none of the decompositions satisfies the properties required of m_1, m_2, n_1 and n_2.