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Problem 10979

Proposed by Donald P. Minassian, Butler University, Indianapolis, IN.

Let f be a function from the open interval (a,b) to a metric space X. When both $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist, but one or both differ from $f(x_0)$, f has a *simple discontinuity* at x_0. Prove that f has at most countably many simple discontinuities.

Solution:

We assume initially that (a,b) is bounded. All neighborhoods are formed within (a,b). Define $S_n = \{x \in (a,b) : \max(|f(x) - \lim_{y\to x^-} f(y)|, |f(x) - \lim_{y\to x^+} f(y)|) > 1/n\}$. We want to show the set of simple discontinuities $S = \bigcup_{n=1}^{\infty} S_n$ is countable. For the sake of contradiction, suppose not. Then at least one of the S_n, say S_m, must be uncountable.

First, we claim that each point $x \in S_m$ is isolated. If not, then $(S_m \setminus \{x\}) \cap (x-\delta, x+\delta) \neq \varnothing$ for arbitrary $\delta > 0$. This implies that either $S_m \cap (x-\delta, x) \neq \varnothing$ or $S_m \cap (x, x+\delta) \neq \varnothing$, and clearly this condition may be asserted separately for at least one of the preceding disjuncts. Therefore, without restriction of generality, suppose we have $y_\delta \in S_m \cap (x-\delta, x)$ for pre-assigned δ. Now either $|f(y_\delta) - \lim_{y\to y_\delta^-} f(y)| > 1/m$ or $|f(y_\delta) - \lim_{y\to y_\delta^+} f(y)| > 1/m$, and it follows that in a neighborhood of y_δ, more specifically $(x-\delta, x)$, there are points x_{lo} and x_{hi} such that $f(x_{hi}) - f(x_{lo}) \geq 1/2m$. Hence the variation of f on an arbitrary left neighborhood $(x-\delta, x]$ of x is at least $1/2m$. This precludes the existence of $\lim_{y\to x^-} f(y)$, which was given, and this contradiction establishes the claim.

Since $x \in S_m$ is isolated, for suitable $k \in \mathbb{N}$, we have $(x - 1/2k, x - 1/2k) \cap S_m = \{x\}$. Define $S_{m,k} = \{x \in S_m | (x - 1/2k, x - 1/2k) \cap S_m = \{x\}\}$. Since $S_m = \bigcup_{k=1}^{\infty} S_{m,k}$, and S_m has been assumed to be uncountable, at least one of the $S_{m,k}$ must likewise be uncountable. But the total length of uncountably many disjoint intervals, each of length $1/k$, is unbounded, contradicting the fact that these intervals are all subsets of (a,b), which has provisionally been assumed to be bounded. This contradiction establishes that each $S_{m,k}$ is countable, which in turn forces each S_n to be countable, and hence ultimately S to be countable.

Applying the preceding result for bounded intervals to intervals of the form $(-n,b)$, (a,n), and $(-n,n)$, we can extend it to $(-\infty,b) = \bigcup_{n=1}^{\infty}(-n,b)$, $(a,\infty) = \bigcup_{n=1}^{\infty}(a,n)$, and $(-\infty,\infty) = \bigcup_{n=1}^{\infty}(-n,n)$, respectively, as countable unions of countable sets. This relaxes the initial assumption that (a,b) be bounded and establishes the general result.

FGCU Problem Group
Florida Gulf Coast University
Ft. Myers, FL
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R. E. Prather, P. P. Ray, R. Richberg (Germany), A. J. Rosenthal and K. A. Fowler, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stephens, R. Stong, X. Wang, J. E. Wilkins, Jr., L. Zacks, FGCU Problem Group, GCHQ Problem Group, NSA Problem Group, and the proposer.

Simple Discontinuities

10979 [2002, 921]. *Proposed by Donald P. Minassian, Butler University, Indianapolis, IN.* Let f be a function from the open interval (a, b) to a metric space X. When both $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist, but one or both differ from $f(x_0)$, f has a *simple discontinuity* at x_0. Prove that f has at most countably many simple discontinuities.

Solution by K. P. Hart, T. U. Delft, Delft, The Netherlands. Write d for the metric on X, write $D = \{x : f \text{ has a simple discontinuity at } x\}$, and write $f(x^-) = \lim_{y\to x^-} f(y)$ and $f(x^+) = \lim_{y\to x^+} f(y)$ whenever these limits exist. The set D is covered by the two sets

$$A = \{x : f(x^-) \text{ exists and } f(x) \neq f(x^-)\},$$
$$B = \{x : f(x^+) \text{ exists and } f(x) \neq f(x^+)\}.$$

For $n \in \mathbb{N}$, put

$$A_n = \{x \in A : d\big(f(x^-), f(x)\big) \geq 2^{-n}\}.$$

We show that each A_n is countable.

We claim that for each $p \in A_n$ there exists $\delta_p > 0$ such that $(p - \delta_p, p) \cap A_n = \emptyset$. Indeed, choose $\delta_p > 0$ so small that $d\big(f(x), f(p^-)\big) < 2^{-n}/3$ for all $x \in (p - \delta_p, p)$. By the triangle inequality, we have $d\big(f(x), f(y)\big) < (2/3)2^{-n}$ when $x, y \in (p - \delta_p, p)$. From this it follows that $d\big(f(x^-), f(x)\big) \leq (2/3)2^{-n}$ if $x \in A \cap (p - \delta_p, p)$ and so indeed $(p - \delta_p, p) \cap A_n = \emptyset$.

Now note that $\{(p - \delta_p, p) : p \in A_n\}$ is a pairwise disjoint family of intervals in the real line, and hence countable, which implies that A_n is countable. But $A = \bigcup_n A_n$, so it follows that A is countable. The same argument shows that B is countable. It follows that $D = A \cup B$ is countable.

Editorial comment. The following references were cited. For the special case $X = \mathbb{R}$: A. Froda, "Sur l'ensemble des discontinuités de première espèce," *Comptes Rendus Acad. Sci. Paris* **186** (1928), 728–729; K. R. Stromberg, *An Introduction to Classical Real Analysis* (Wadsworth, 1981), exercise 3, p. 131; W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, 1976), exercise 4.17, p. 100.

For general metric spaces: G. Choquet, *Cours d'analyse* (Masson, 1964), p. 142.

For a still stronger result: Y.-M. Wong, "A Theorem on Points of Discontinuity of Functions," *J. London Math. Soc.* **40** (1965), 324–325.

Similar results: M. Eşanu, "Une généralisation des théorèmes de A. Froda et Yu. Y. Prokhorov concernant les points de discontinuité de la première espèce," *Stud Cerc. Mat.* **44** (1992), 297–299.

Also solved by S. Amghibech (Canada), M. Bataille (France), J. Boersema, M. W. Botsko, P. Budney, R. Chapman (U. K.), J. Cobb, D. Donini (Italy), P. J. Fitzsimmons, J. Grivaux (France), G. L. Isaacs, P. Kordulova (Czech Republic), O. P. Lossers (The Netherlands), K. McInturff, M. D. Meyerson, A. Mohammed, G. Muller, A. Nijenhuis, V. Pambuccian, C. Petalas & T. Vidalis, K. A. Ross, J. Rouse, A. W. Schurle (Saudi Arabia), A. Strenger, R. Stephens, R. Stong, X. Wang, L. Zhou, FGCU Problem Group, GCHQ Problem Group, Univ. Louisiana–Lafayette Math Club, and the proposer.