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Problem 10978

Proposed by Jean-Pierre Grivaux, Lycée Chaptal, Paris, France.

Let $P_n(x) = x^{n^2+n-1} - 3x^{n^2} + x^{n^2-1} + x^{n^2-n} + x^{2n-1} + x^n - 3x^{n-1} + 1$. Show for every integer $n \geq 2$ and real $x \geq 0$ that $P_n(x) \geq 0$.

Solution:

We note that $P_n(1) = 0$, hence $(x-1)$ is a factor of $P_n(x)$. After some routine differentiations and a little algebra, we also determine that $P_n'(1) = P_n''(1) = P_n'''(1) = 0$, which indicates that $(x-1)^4$ is a factor of $P_n(x)$. For $n \geq 2$, $P_n(x)$ has four sign changes, so by Descartes' Rule the number of positive roots can be at most four, all of which are accounted for by the root of multiplicity four at $x = 1$. Since $P_n(0) = 1$, it follows that the only root of $P_n(x)$ on $[0, \infty)$ occurs at $x = 1$. Observing that $P_n(x)$ is positive for large x, as x^{n^2+n-1} eventually dominates, it is apparent that $P_n(x) \geq 0$ whenever $x \geq 0$.

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Solution I by Julio Kuplinsky, Montclair, NJ. We will show that $\phi_n(t) = \prod_{i=1}^{n-1} \sum_{j=0}^{i} t^j$ by proving that $\phi_n(t) = \phi_{n-1}(t) \sum_{j=0}^{n-1} t^j$ for $n \geq 2$, since $\phi_1(t) = 1$ is obvious. For $n \geq 2$, the disorder of a permutation $\pi \in S_n$ having $\pi_i = 1$ is $i - 1 + \operatorname{dis}(\sigma)$, where

$$\sigma = (\pi_{i+1} - 1, \pi_{i+2} - 1, \ldots, \pi_{i+n-1} - 1) \in S_{n-1},$$

with indices understood modulo n. This yields a bijection to S_{n-1} from S_n^i, where $S_n^i = \{\pi \in S_n : \ \pi_i = 1\}$. Thus

$$\phi_n(t) = \sum_{i=1}^{n} \sum_{\pi \in S_n^i} t^{\operatorname{dis}(\pi)} = \sum_{i=1}^{n} \sum_{\sigma \in S_{n-1}} t^{i-1+\operatorname{dis}(\sigma)} = \left(\sum_{i=1}^{n} t^{i-1} \right) \phi_{n-1}(t).$$

Solution II by Tina Garrett, Carleton College, Northfield, MN. Given a permutation π, let a_i be the number of elements skipped before removing i after the most recent (if any) removal. Thus $\sum_{i=1}^{n} a_i = \operatorname{dis}(\pi)$. The numbers a_i with $0 \leq a_i \leq n - i$ determine the placement of successive elements in π, so we have a bijection from S_i to $\{(a_1, \ldots, a_n) : \ 0 \leq a_i \leq n - i\}$. Thus

$$\sum_{\pi \in S_n} t^{\operatorname{dis}(\pi)} = \left(\sum_{a_1=0}^{n-1} t^{a_1} \right) \left(\sum_{a_2=0}^{n-2} t^{a_2} \right) \left(\sum_{a_3=0}^{n-3} t^{a_3} \right) \cdots \left(\sum_{a_n=0}^{0} t^{a_n} \right) = \prod_{i=1}^{n-1} \sum_{j=0}^{i} t^j.$$

Editorial comment. The generating function is the same as the generating function for permutations by number of inversions, and the proof in Solution II gives a direct explanation of why. Furthermore, the disorder of a permutation is directly related to the major index of its inverse, where the *major index* of a permutation $(\pi_1, \ldots, \pi_n)$ is the sum of those i such that $a_i > a_{i+1}$. See R. P. Stanley's *Enumerative Combinatorics, Vol. I* (Cambridge University Press, 1997) for a discussion of permutation statistics.

Also solved by D. Beckwith, J. C. Binz (Switzerland), D. Callan, R. Chapman (U. K.), J. Clark & W. P. Wardlaw, P. Corn, K. Laghate & M. N. Deshpande (India), G. Lavau (France), S. C. Locke, O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), M. A. Prasad (India), W. R. Smythe, J. H. Steelman, R. Stong, J. T. Ward, W.-J. Woan & D. Hough, M. Woltermann, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Nonnegative Polynomials

10978 [2002, 921]. *Proposed by Jean-Pierre Grivaux, Lycée Chaptal, Paris, France.* Let $P_n(x) = x^{n^2+n-1} - 3x^{n^2} + x^{n^2-1} + x^{n^2-n} + x^{2n-1} + x^n - 3x^{n-1} + 1$. Show that for every integer $n \geq 2$ and real $x \geq 0$, $P_n(x) \geq 0$.

Solution by Tsehaye Andebrhan, High School, Asmara, Eritrea. Note that $P_n(1) = P_n'(1) = P_n''(1) = P_n'''(1) = 0$. Therefore, $P_n(x)$ has a quadruple root at $x = 1$. The coefficients of $P_n(x)$ have four sign changes; therefore, by Descartes's Rule of Signs, $P_n(x)$ has at most four positive real roots counted with multiplicity. This leaves no room for any further positive zeros of P_n. Since the leading coefficient of $P_n(x)$ is positive, it follows that $P_n(x) \geq 0$ when $x \geq 0$.

Also solved by S. Amghibech (Canada), W. W. S. Au, R. Barbara (Lebanon), M. Bataille (France), J. M. Benedict, D. Callan, R. Chapman (U.K.), C. Curtis, K. Dale (Norway), L. M. DeAlba, D. Donini (Italy), Y. Dumont (France), A. Eydelzon (Israel), M. Getz and D. Jones, C. R. Hampton, D. Henderson, T. Hermann, G. L. Isaacs, W. Janous (Austria), J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, A. Nijenhuis,

R. E. Prather, P. P. Ray, R. Richberg (Germany), A. J. Rosenthal and K. A. Fowler, N. C. Singer, A. Stadler (Switzerland), A. Stenger, R. Stephens, R. Stong, X. Wang, J. E. Wilkins, Jr., L. Zacks, FGCU Problem Group, GCHQ Problem Group, NSA Problem Group, and the proposer.

Simple Discontinuities

10979 [2002, 921]. *Proposed by Donald P. Minassian, Butler University, Indianapolis, IN.* Let f be a function from the open interval (a, b) to a metric space X. When both $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist, but one or both differ from $f(x_0)$, f has a *simple discontinuity* at x_0. Prove that f has at most countably many simple discontinuities.

Solution by K. P. Hart, T. U. Delft, Delft, The Netherlands. Write d for the metric on X, write $D = \{x\colon f \text{ has a simple discontinuity at } x\}$, and write $f(x^-) = \lim_{y\to x^-} f(y)$ and $f(x^+) = \lim_{y\to x^+} f(y)$ whenever these limits exist. The set D is covered by the two sets

$$A = \{x\colon f(x^-) \text{ exists and } f(x) \neq f(x^-)\},$$
$$B = \{x\colon f(x^+) \text{ exists and } f(x) \neq f(x^+)\}.$$

For $n \in \mathbb{N}$, put

$$A_n = \{x \in A\colon d\big(f(x^-), f(x)\big) \geq 2^{-n}\}.$$

We show that each A_n is countable.

We claim that for each $p \in A_n$ there exists $\delta_p > 0$ such that $(p - \delta_p, p) \cap A_n = \emptyset$. Indeed, choose $\delta_p > 0$ so small that $d\big(f(x), f(p^-)\big) < 2^{-n}/3$ for all $x \in (p - \delta_p, p)$. By the triangle inequality, we have $d\big(f(x), f(y)\big) < (2/3)2^{-n}$ when $x, y \in (p - \delta_p, p)$. From this it follows that $d\big(f(x^-), f(x)\big) \leq (2/3)2^{-n}$ if $x \in A \cap (p - \delta_p, p)$ and so indeed $(p - \delta_p, p) \cap A_n = \emptyset$.

Now note that $\{(p - \delta_p, p)\colon p \in A_n\}$ is a pairwise disjoint family of intervals in the real line, and hence countable, which implies that A_n is countable. But $A = \bigcup_n A_n$, so it follows that A is countable. The same argument shows that B is countable. It follows that $D = A \cup B$ is countable.

Editorial comment. The following references were cited. For the special case $X = \mathbb{R}$: A. Froda, "Sur l'ensemble des discontinuités de première espèce," *Comptes Rendus Acad. Sci. Paris* **186** (1928), 728–729; K. R. Stromberg, *An Introduction to Classical Real Analysis* (Wadsworth, 1981), exercise 3, p. 131; W. Rudin, *Principles of Mathematical Analysis* (McGraw-Hill, 1976), exercise 4.17, p. 100.

For general metric spaces: G. Choquet, *Cours d'analyse* (Masson, 1964), p. 142.

For a still stronger result: Y.-M. Wong, "A Theorem on Points of Discontinuity of Functions," *J. London Math. Soc.* **40** (1965), 324–325.

Similar results: M. Eşanu, "Une généralisation des théorèmes de A. Froda et Yu. Y. Prokhorov concernant les points de discontinuité de la première espèce," *Stud Cerc. Mat.* **44** (1992), 297–299.

Also solved by S. Amghibech (Canada), M. Bataille (France), J. Boersema, M. W. Botsko, P. Budney, R. Chapman (U. K.), J. Cobb, D. Donini (Italy), P. J. Fitzsimmons, J. Grivaux (France), G. L. Isaacs, P. Kordulova (Czech Republic), O. P. Lossers (The Netherlands), K. McInturff, M. D. Meyerson, A. Mohammed, G. Muller, A. Nijenhuis, V. Pambuccian, C. Petalas & T. Vidalis, K. A. Ross, J. Rouse, A. W. Schurle (Saudi Arabia), A. Strenger, R. Stephens, R. Stong, X. Wang, L. Zhou, FGCU Problem Group, GCHQ Problem Group, Univ. Louisiana–Lafayette Math Club, and the proposer.