American Mathematical Monthly

Problem 10916

Proposed by Gertrude Ehrlich, University of Maryland, College Park, MD

Available are two beakers A and B, having volumes a and b liters, respectively, a source of water, and a drain. Water may be poured into the beakers from the source, or from each other, either filling the receiving beaker or emptying the source beaker, and the beakers may be emptied into the drain. Using only these operations, show that if a and b are relatively prime positive integers, then for every integer m with $1 \le m \le b$ *it is possible to reach a state in which beaker B contains m liters.*

Solution:

We may assume $b > 1$, otherwise fill B with one liter and we are done. Since $(a,b) = 1$, we have non-zero integers x,y such that $xa + yb = 1$. Then $xa \equiv 1 \bmod b$, and if r is the residue of $x \bmod b$, we have $ra \equiv 1 \bmod b$ with $1 \le r < b$. Finally, we have $mra \equiv m \bmod b$. Consider a volume consisting of mra liters. This can be delivered to B by filling A a total of mr times and pouring each full beaker A into B, interrupting, if necessary, the emptying of A by emptying B each time it is exactly filled by this process, and continuing to pour the remaining contents of A into B in this manner before refilling A from the (necessarily infinite) source. When the mra^{th} liter of water has been poured into B, the amount remaining in B will be m liters. This is clear, since the successive filling and emptying of B discards a multiple of b liters, and the amount left over from a starting volume of mra liters would be $mra \bmod b$, which is m by the above remarks. In the event that $m = b$, just top off B directly from the source.

FGCU Problem Group
2-21-02

where $q(x) = \sum_{j=1}^{p-1} jx^{p-1-j}$. Hence, $a_p(x)\Phi_n(x) - q(x^{rm})(x^{rm} - 1) = p$. Since $x^{rm} - 1$ is divisible by $\Phi_m(x)$, there are integer polynomials a_p and b_p such that $a_p\Phi_n(x) + b_p\Phi_m(x) = p$. If n/m is divisible by distinct primes p and q, then there exist integers c and d such that $pc + qd = 1$. It follows that $(ca_p + da_q)\Phi_n(x) + (cb_p + db_q)\Phi_m(x) = 1$. Therefore, $k = 1$.

It remains to show that if $n = mp^j$, then $k = p$. By the preceding paragraph, $k \le p$, so we need show only that $k \ge p$. If ζ is a primitive (rs)th root of unity, then ζ^s is a primitive rth root of unity. Hence $\Phi_{rs}(x)$ divides $\Phi_r(x^s)$. In particular, $\Phi_n(x)$ divides $\Phi_m(x^{p^j})$. Let $\overline{f}$ denote the reduction of f modulo p, so that $\overline{f}$ is a polynomial over $\mathbb{F}_p$. We have shown that $\overline{\Phi_n}(x)$ divides $\overline{\Phi_m}(x^{p^j})$, which equals $(\overline{\Phi_m}(x))^{p^j}$.

Now let a and b be integer polynomials such that $a\Phi_n + b\Phi_m = k$. It follows that $\overline{a}\overline{\Phi_n} + \overline{b}\overline{\Phi_m} = \overline{k}$, so any irreducible divisor u of $\overline{\Phi_m}$ is also a divisor of $\overline{\Phi_n}$, hence of $\overline{k}$. Since k is a natural number, the only way this can occur over $\mathbb{F}_p$ is if $k \equiv 0 \pmod{p}$. Therefore, $k \ge p$, as desired.

Also solved by Robin Chapman (U. K.), C. P. Rupert, the GCHQ Problem Solving Group (U. K.), and the proposer.

The Jug Problem with a Source and a Sink

10916 [2002, 77]. *Proposed by Gertrude Ehrlich, University of Maryland, College Park, MD.* Available are two beakers A and B, having volumes a liters and b liters, respectively, a source of water, and a drain. Water may be poured into the beakers from the source or from each other, either filling the receiving beaker or emptying the source beaker, and beakers may be emptied into the drain. Using only these operations, show that if a and b are relatively prime positive integers, then for every integer m with $1 \le m \le b$ it is possible to reach a state in which beaker B contains m liters.

Solution by Reiner Martin, New York, NY. There exists a positive integer k with $ka \equiv m \pmod{b}$. For k times, fill A from the source and empty it into B, emptying B into the drain whenever it is full. At the end, B contains ka liters minus some integer multiple of b, and so it must contain m liters.

Editorial comment. Thomas J. Pfaff notes that the paper "The Generalized Jug Problem," which he coauthored with Max M. Tran, will appear in the *Journal of Recreational Mathematics*, and H. F. Mattson Jr. points out that it appears on pages 237-239 of his text *Discrete Mathematics with Applications*, (Wiley, 1993). Several students at Skidmore College remarked that one case of this problem played a role in the film *Die Hard with a Vengeance*, where Bruce Willis and Samuel L. Jackson were able to solve it just in the nick of time.

Also solved by J. Alfano, M. R. Avidon, M. Bowron, E. F. Boyer, J. Brawner, R. Chapman (U. K.), J. Christopher, W. R. Boyd (France), H. Y. Far, E. Grant, J.-P. Grivaux (France), D. Hancock, M. Hildebrand, S. Kaczkowski, N. Komanda, H. Kwong, A. Lenskold, J. H. Lindsey II, O. P. Lossers (Netherlands), H. F. Mattson, Jr., A. Naklash, C. Palmer, N. Passell, R. Patenaude, D. Pecota, T. J. Pfaff, A. Rand, D. Seff, P. Schumer, O. e Silva (Portugal), J. H. Steelman, R. Stong, C. N. Swanson, D. G. Treat, D. I. Velleman, J. T. Ward, L. Zhou, two anonymous solvers, FGCU Problem Group, GCHQ Problem Solving Group (U. K.), LSU Problems Solving Group, NSA Problems Group, Skidmore College Problem Group, and the proposer

A Congruence Involving Ceilings

10918 [2002, 77]. *Proposed by Mathias Beck, State University of New York, Binghampton, NY.* Prove that for all positive integers a and b,

$$a + (-1)^b \sum_{m=0}^{a} (-1)^{\lceil bm/a \rceil} \equiv b + (-1)^a \sum_{n=0}^{b} (-1)^{\lceil an/b \rceil} \pmod{4}.$$