American Mathematical Monthly

Problem 10830

Proposed by Floor van Lamoen, Goes, The Netherlands

A triangle is divided by its three medians into six smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

Solution:

Figure 1 is an illustration of the theorem. The darker circle is the circle common to the six circumcenters.

We offer an analytic proof. First we develop a formula for the center of the circumcircle of an arbitrary triangle. The perpendicular bisectors of the sides of a triangle are concurrent at the circumcenter, so any two determine this point. Any triangle in the xy plane has (at least) two sides that are not horizontal, so the perpendicular bisectors of those sides have defined slope. Refer to Figure 2 for notation. Without loss of generality, we may assume $\overline{XY}$ and $\overline{XZ}$ are not parallel to the x-axis, hence $b \neq d$ and $b \neq f$. The perpendicular bisector of $\overline{XY}$ has equation

$$y = \frac{b+d}{2} + \frac{a-c}{d-b}\left(x - \frac{a+c}{2}\right)$$

and perpendicular bisector of XZ has equation

$$y = \frac{b+f}{2} + \frac{a-e}{f-b}\left(x - \frac{a+e}{2}\right)$$

Equating these two yields the following co-ordinates for the center of the circumcircle of $\triangle XYZ$:

$$(\alpha) \qquad x_{cen} = -\frac{1}{2}\frac{d^2f-d^2b+db^2-a^2f+c^2f-c^2b-df^2+bf^2-fb^2+a^2d-e^2d+e^2b}{af-cf+cb-ad+ed-eb}$$

$$(\beta) \qquad y_{cen} = \frac{1}{2}\frac{cb^2-eb^2+c^2e+ae^2-ce^2+ed^2-a^2e-ad^2-ac^2+af^2-cf^2+a^2c}{af-cf+cb-ad+ed-eb}$$

Now consider an arbitrary triangle and its six medial subtriangles. Refer to Figure 3 for notation. The numbers 1 thru 6 identify the subtriangles. For definiteness, orient the triangle so that the centroid is at the origin and AC is horizontal.

Again, without restriction of generality, we will determine the circumcenter of the triangle formed by the circumcenters of subtriangles 1, 2, and 3. Call this point (x_{123}, y_{123}). Then we will repeat this calculation for subtriangles 2, 3, and 4, finding (x_{234}, y_{234}). We claim that the theorem is proved if $(x_{123}, y_{123}) = (x_{234}, y_{234})$.

For certainly the circumcenters of subtriangles 2 and 3 are common to both calculations, and hence the radii of the circumcircles centered at (x_{123}, y_{123}) and (x_{234}, y_{234}) agree. Likewise, the labeling of vertices in Figure 3 is arbitrary, so we may conclude that $(x_{234}, y_{234}) = (x_{345}, y_{345})$ and finally $(x_{345}, y_{345}) = (x_{456}, y_{456})$, again with the radii equal.

Using formulas α and β, and writing (x_i, y_i) for the circumcenter of the i^{th} subtriangle, we get:

$$(x_1, y_1) = \left(\frac{1}{2}\frac{3s^2+2rt+t^2}{r-t}, -\frac{1}{4}\frac{r^3+3r^2t+2rt^2+7rs^2-s^2t}{s(r-t)}\right)$$

$$(x_2, y_2) = \left(-\frac{1}{2}\frac{3s^2+r^2+2rt}{r-t}, \frac{1}{4}\frac{3rt^2+t^3t+2r^2t-rs^2+7s^2t}{s(r-t)}\right)$$

$$(x_3, y_3) = \left(\frac{1}{4} \frac{3s^2+2r^2+t^2}{r-t}, -\frac{1}{4} \frac{2s^2t+rs^2+rt^2+2r^2t}{s(r-t)} \right)$$

$$(x_4, y_4) = \left(\frac{3}{4}r + \frac{1}{4}t, -\frac{1}{4} \frac{r^2+rt-2s^2}{s} \right)$$

It is clear that all denominators above are non-zero. Now we compute (x_{123}, y_{123}) and (x_{234}, y_{234}) explicitly, again using formulas α and β:

$$x_{123} = -\frac{1}{8} \frac{r^4t+2t^2r^3+6r^2s^2t+2t^3r^2+3s^4r+6rs^2t^2+t^4r+3ts^4}{s^2(r-t)^2}$$

$$x_{234} = -\frac{1}{8} \frac{r^4t+2t^2r^3+6r^2s^2t+2t^3r^2+3s^4r+6rs^2t^2+t^4r+3ts^4}{s^2(r-t)^2}$$

$$y_{123} = -\frac{1}{8} \frac{r^4+2r^3t+4r^2s^2+4s^2tr+2rt^3+t^4+6s^4+4s^2t^2}{s(r-t)^2}$$

$$y_{234} = -\frac{1}{8} \frac{r^4+2r^3t+4r^2s^2+4s^2tr+2rt^3+t^4+6s^4+4s^2t^2}{s(r-t)^2}$$

Clearly $(x_{123}, y_{123}) = (x_{234}, y_{234})$, hence the claim is established and the theorem proved.

The resulting computation for the expectation is

$$\mathrm{E}(X_i) = \frac{k}{n}\frac{1+k}{2} + \frac{n-k}{n}\left(i + k\frac{n-i}{n-1}\right),$$

which simplifies to the expression initially claimed.

Editorial comment. M.N. Deshpande conjectured that $\sum_{i=1}^{n} E(X_i^r) = \sum_{i=1}^{n} i^r$. This holds trivially for $r = 1$, and Deshpande proved it for $r = 2$.

The permutation π can be viewed as the "picking order"; the n teams have equal chance to pick among the first k. The remaining team with the worst record picks next, and so on. For further information about the use of this and other schemes by the NBA up to 1995, see Y. Gerchak, H. E. Mausser, and M. J. Magazine, "The Evolution of Draft Lotteries in Professional Sports: Back to Moral Hazard?", *Interfaces* 25 (1995) 30–38. The proposer considered other lottery schemes in S. G. Penrice, "Applying Elementary Probability Theory to the NBA Draft Lottery, *SIAM Review* 37 (1995) 598–602.

Also solved by D. Beckwith, B. Burdick, R. Chapman (U. K.), M. N. Deshpande (India), J. T. Lewis, J. H. Lindsey, J. H. Nieto (Venezuela), S. Riccarelli (Italy), W. R. Smythe, R. Stong, L. Zhou, Theory First, and the proposer.

Circumcenters on a Circle

10830 [2000, 863]. *Proposed by Floor van Lamoen, Goes, The Netherlands.* A triangle is divided by its three medians into 6 smaller triangles. Show that the circumcenters of these smaller triangles lie on a circle.

Solution by the editors. The medians of any triangle fit together to form a triangle without changing their orientations. We will need the similar triangle made from segments half the lengths of the medians; we call this the *semi-median triangle.* Let d be the diameter of the circumcircle of the semi-median triangle.

The medians of a triangle meet at its centroid and divide each other into two segments, one twice as long as the other. Let the perpendicular bisectors of these six segments be $L_1, \dots, L_6$, cyclically. The intersections of consecutive pairs in this cycle are the circumcenters that we must show are cocircular. Let O_i be the intersection of L_i and L_{i+1} (all operations on indices are modulo 6). The six points $O_1, \dots, O_6$ form a hexagon that need not be convex or simple.

The perpendicular bisectors are parallel in pairs, with L_i is parallel to L_{i+3} for all i. Hence $O_1O_2O_4O_5$ is a trapezoid, as are $O_2O_3O_5O_6$ and $O_3O_4O_6O_1$. We show that these three trapezoids are isosceles by showing that their three shared "diagonals" O_1O_4, O_2O_5, and O_3O_6 all have length d. We then show that the hexagon is circumscribable by a circle whose center is the circumcenter of the triangle formed by the midpoints of the three segments O_1O_4, O_2O_5, and O_3O_6.

By symmetry, it suffices to show that the length of O_1O_4 is d. Let M_i be the line parallel to and halfway between L_i and L_{i+3}, for $i \in \{1, 2, 3\}$. Let a_i be the segment perpendicular to and joining L_i and L_{i+3} that lies along a median of the original triangle. Since the segments a_1, a_2, a_3 are each half the length of a median, they fit together to form the semi-median triangle.

We now form two additional copies of the semi-median triangle. To make the first, let segments congruent to a_1 and a_2 be created at O_1 by sliding a_1 and a_2 along M_1 and M_2, respectively. The segment determined by the ends of these segments opposite to O_1 is congruent to a_3, forming a semi-median triangle. The second triangle is made in the same way using O_4 instead of O_1. These two triangles both have circumcenter at the intersection P_1 of M_1 and M_2. Also, O_1 and O_4 lie on the common circumcircle and are opposite ends of a diameter. Hence the length of O_1O_4 is d, as claimed, and P_1 is the midpoint of O_1O_4.

For $i \in \{1, 2, 3\}$, we have P_i as the intersection of M_i and M_{i+1} and the midpoint of O_iO_{i+4}. Consider the circumcenter of the triangle $P_1P_2P_3$; it is the intersection of the perpendicular bisectors of the sides. Since P_1P_2 is the midline of trapezoid $O_1O_2O_4O_5$, its perpendicular bisector is also the perpendicular bisector of O_1O_2 and O_4O_5 (chords of the hexagon) and hence contains the center of the circumscribing circle. The same result holds for P_2P_3 and P_3P_1.

Editorial comment. The proposer observed that Clark Kimberling, in his paper "Triangle Centers and Central Triangles", *Congressus Numerantium,* **129** (1998), states that the circumcenters in question are circumscribable by a conic. The problem here was to show that this conic is in fact always a circle.

P. Y. Woo submitted a proof of a converse of the theorem: For each point P inside a triangle, consider the six triangles into which the triangle is divided by the three cevians through P. If the circumcenters of these six triangles lie on a circle, then P is the centroid of the triangle and the cevians are its medians.

The submitted solutions used analytic geometry (or complex numbers) and involved lengthy computations (some done with Maple or Mathematica). The editors felt that a coordinate-free statement deserves a coordinate-free solution; such a solution may shed more light on why the result is true.

Solved also by S. Amghibech (France), J. Anglesio (France), M. Benedicty, I. Dimitric, J.-P. Grivaux (France), N. Komanda, K.-W. Lau (Hong Kong), K. Y. Li (China), J. H. Lindsey, A. Sasane (The Netherlands), E. R. Scheinerman, A. Vorobyov, M. Woltermann, P. Y. Woo (Hong Kong), Florida Gulf Coast Problems Group, GCHQ Problem Solving Group (U.K.), and the proposer.

When ABC is Equilateral

10838 [2000, 950]. *Proposed by Florian S. Pârvânescu, Slatina, Romania.* Let M be any point in the interior of triangle ABC, and let D, E, and F be points on the perimeter such that AD, BE, and CF are concurrent at M. Show that if the triangles BMD, CME, and AMF all have equal areas and equal perimeters, then ABC is equilateral.

Solution by Roy Barbara, Lebanese University, Ranar, Labanon. Write $\mathcal{A}(\cdot)$ for area and $P(\cdot)$ for perimeter.

We first show that M is the centroid of $\triangle ABC$. Set $\mathcal{A}(BMD) = \mathcal{A}(CME) = \mathcal{A}(AMF) = u$, $\mathcal{A}(AME) = x$, $\mathcal{A}(CMD) = y$, and $\mathcal{A}(BMF) = z$. Now

$$\frac{x}{u} = \frac{EA}{EC}, \quad \frac{y}{u} = \frac{DC}{DB}, \quad \frac{z}{u} = \frac{FB}{FA}.$$

This and Ceva's theorem yield $xyz = u^3$. Now $\frac{\mathcal{A}(ADC)}{\mathcal{A}(ADB)} = \frac{DC}{DB} = \frac{y}{u}$, so

$$\frac{x+u+y}{z+2u} = \frac{y}{u} = \frac{x+u+y-y}{z+2u-u} = \frac{x+u}{z+u}.$$

Therefore, $xu + u^2 = yu + yz$. Replacing yz with u^3/x and dividing by u yields $x + u = y + u^2/x$. Similarly, $y + u = z + u^2/y$ and $z + u = x + u^2/z$. Summing and dividing by u^2 yields $1/x + 1/y + 1/z = 3/u$.

Subject to this, the product $1/xyz$ is maximized when and only when $1/x = 1/y = 1/z = 1/u$, with the maximum being $1/u^3$. Since $xyz = u^3$, we thus obtain $1/x = 1/y = 1/z$. Thus $x = y = z$, and M is the centroid of $\triangle ABC$.