American Mathematical Monthly

Problem 10809 - May 2000

Proposed by David Beckwith, Sag Harbor, NY.
For $|x| < 1$*, prove that* $\sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}$

Solution:

We note that both series converge absolutely for the specified values of x, hence we are free to

rearrange the terms in either sum. Writing $\frac{1}{1-x^n} = 1 + x^n + x^{2n} + \cdots = \sum_{k=1}^{\infty} x^{n(k-1)}$ and $\frac{1}{1-x^{2n}} = 1 + x^{2n} + x^{4n} + \cdots = \sum_{k=1}^{\infty} x^{2n(k-1)}$, we transform the two given series respectively:

(1) $\sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n} = \sum_{n=1}^{\infty} x^{n(n+1)/2} \sum_{k=1}^{\infty} x^{n(k-1)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^{n(n+1)/2+n(k-1)}$ and
(2) $\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} = \sum_{n=1}^{\infty} x^n \sum_{k=1}^{\infty} x^{2n(k-1)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^{n+2n(k-1)} = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} x^{s+2s(t-1)}$,

where the dummy indices in the second double series are changed for subsequent clarity.

Claim: The map $\phi : \mathbb{N}^2 \to \mathbb{N}^2$, $(n,k) \mapsto (s,t)$ defined for even n by $s = \frac{n}{2}$ and $t = k + \frac{n}{2}$

and for odd n by $s = k + \frac{n-1}{2}$ and $t = \frac{n+1}{2}$ is bijective and has the property that

$n(n+1)/2 + n(k-1) = s + 2s(t-1)$.

Surjectivity of ϕ is clear. To establish injectivity, write $(s',t') = \phi(n',k')$ and suppose $(s,t) = (s',t')$.

We examine the three cases:

(i) n and n' even - then $s = s' \Rightarrow \frac{n}{2} = \frac{n'}{2} \Rightarrow n = n'$, whence $t = t' \Rightarrow k + \frac{n}{2} = k' + \frac{n'}{2} \Rightarrow k = k'$

(ii) n and n'odd - then $t = t' \Rightarrow \frac{n+1}{2} = \frac{n'+1}{2} \Rightarrow n = n'$, whence $s = s'$ $\Rightarrow k + \frac{n-1}{2} = k' + \frac{n'-1}{2} \Rightarrow k = k'$

(iii) n even and n' odd - then $s = s' \Rightarrow \frac{n}{2} = k' + \frac{n'-1}{2}$ and $t = t' \Rightarrow k + \frac{n}{2} = \frac{n'+1}{2}$, and after some algebra we find $k + k' = 1$, which is impossible.

The two viable cases show ϕ is injective, and hence a bijection. We now establish the property that for $(s,t) = \phi(n,k)$, $n(n+1)/2 + n(k-1) = s + 2s(t-1)$.

There are two cases, odd and even. The even case follows directly by substitution,

$s + 2s(t-1) = \frac{n}{2} + 2(\frac{n}{2})(k + \frac{n}{2} - 1) = \frac{n^2}{2} + nk - \frac{n}{2} =$
$\frac{n}{2}(n-1) + nk + n - n = \frac{n(n+1)}{2} + n(k-1)$,
as does the odd case, for which we omit the detailed calculation.

This establishes our claim, and it follows immediately that

$\sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^{n(n+1)/2+n(k-1)} = \sum_{s(n,k)} \sum_{t(n,k)} x^{s+2s(t-1)} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x^{n+2n(k-1)} =$
$\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}$, as required.

For fixed n, every polynomial is a linear combination of the polynomials $(u-n)_{(m)}$ for $m \geq 0$. By linearity, $F(u_{(n)}p(u)) = F(p(u+n))$ for every polynomial p. In particular,

$$F(u_{(n)}u^h) = F((u+n)^h). \tag{2}$$

The binomial theorem and $F(u^j) = B_j$ turn the right side of (2) into the right side of the desired identity. The left side of (2) becomes the left side of the desired identity by application of (1), the substitution $j = n-k$, and $F(u^{n-j+h}) = B_{n-j+h}$.

Editorial comment. Define the "exponential polynomials" $\varphi_n(a)$ by $\varphi_n(a) = \sum_k \left\{ {n \atop k} \right\} a^k$, where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind, so that $\varphi_n(1) = B_n$. By the same method, Cigler proved more generally that

$$\sum_{k=1}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \varphi_{k+h}(a) = a^n \sum_{j=0}^{h} \binom{h}{j} \varphi_j(a) n^{h-j}.$$

and gave a q-analogue using the Gould-Carlitz q-Stirling numbers.

Solved also by D. Beckwith, S. Cautis (Canada), R. Chapman (U. K.), O. P. Lossers (The Netherlands), J. H. Nieto (Venezuela), H.-J. Seiffert (Germany), R. Stong, GCHQ Problems Group (U. K.), and the proposer.

An Identity for Partitions of Integers

10809 [2000, 566]. *Proposed by David Beckwith, Sag Harbor, NY.* For $|x| < 1$, prove that

$$\sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}.$$

Composite solution by Karl David, Milwaukee School of Engineering, Milwaukee, WI, and the Assumption College Problems Group, Worcester, MA. Let $f(x)$ denote the series on the left. Separating the terms for even and odd n on the left and using the expansion of the geometric series yields

$$f(x) = \sum_{k=1}^{\infty} \frac{x^{k(2k-1)}}{1-x^{2k-1}} + \sum_{n=1}^{\infty} \frac{x^{n(2n+1)}}{1-x^{2n}} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} x^{(2k-1)n} + \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty} x^{(2k-1)n}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n} x^{(2k-1)n} + \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty} x^{(2k-1)n} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} x^{(2k-1)n} = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}}.$$

The series manipulations are justified by absolute convergence for $|x| < 1$.

Editorial comment. George Andrews noted that the problem appears on p. 28 of P. A. MacMahon's *Combinatory Analysis*, vol. 2, which provides several additional expressions for the series.

Solved also by S. Amghibech (France), G. E. Andrews, J. Anglesio (France), P. K. Badri (India), M. Bataille (France), D. Bizzarri (Belgium), R. Chapman (U. K.), J. Cigler (Austria), L. Clark, K. Dale (Norway), J. E. Dawson (Australia), P. Deiermann, D. Donini (Italy), A. Goddijn (The Netherlands), J.-P. Grivaux (France), T. Hermann, M. Hoffman, P. Hohler (Switzerland), W. Janous (Austria), D. Jiang, F. K. Kenter, R. A. Kopas, C. Krattenthaler (Austria), H. Kwong, K.-W. Lau (China), J. H. Lindsey, O. P. Lossers (The Netherlands), B. A. Lotto, S. Marivani, R. Martin, H. F. Mattson, G. McGuire, K. McInturff, W. W. Meyer, W. Moser (Canada), H. Müller (Germany), S. Namli (Turkey), D. Neuenschwander (Switzerland), S. Northshield, T. J. Osler, S. Pedersen (Denmark), D. P. Peter (Hungary), M. A. Pinsky, H. Prodinger (Austria), O. G. Ruehr,

C. P. Rupert, V. Schindler (Germany), W. J. Seaman, J. H. Smith, J. Stadler, A. Stadler (Switzerland), H. A. Steinberg, A. Stenger, R. Stong, P. Szeptycki, C. L. Vanden Eynden, J. T. Ward, R. Weinstock, X. Wen, C. Y. Yildirim (Turkey), L. Zhou, Centenary Problems Group, Con Amore Problems Group (Denmark), Florida Gulf Coast Problems Group, GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.

The Distance Between the Endpoints of a Longest Path

10811 [2000, 566]. *Proposed by Proposed by Phil Tracy, Liverpool, NY.* Let G be a simple graph whose longest path has ends x and y and has length l. Let s be the sum of the degrees of x and y. Show that the distance from x to y (the length of the shortest path from x to y) is at most $\max\{l-s+2, 2\}$.

Solution by NCCU Problems Group, North Carolina Central University, Durham, NC. More generally, let P be any maximal path in G (that is, P is a path that is not a subpath of a longer path). Let l be the length of P. Let x, y be its endpoints, let s be the sum of their degrees, and let $d(x, y)$ be the distance between them. We prove that $d(x, y) \le \max\{l-s+2, 2\}$.

We may assume that $d(x, y) > 2$. Let $V = \{v_1, \ldots, v_{l-1}\}$ be the set of vertices between x and y on P, in order. By the maximality of P, every neighbor of x or y is in V. Let v_q and v_r be neighbours of x and y, respectively, with $|q-r|$ minimal. No vertex between v_q and v_r on P is a neighbor of either endpoint. Since x and y have no common neighbor and G is simple, exactly $l-1-s$ vertices in V are neighbors of neither endpoint. Hence $|q-r| \le l-s$, and $d(x, y) \le l-s+2$, as required.

Editorial comment. Properties of the expression $d(x, y) + d(x) + d(y)$ were also studied in Problem 10647 [1998, 176; 2001, 470] of this *Monthly*.

Solved also by D. Beckwith, S. Cautis (Canada), R. J. Chapman (U. K.), L. S. Chandran & L. S. Ram (India), M. M. Cropper, P. P. Dalyay (Hungary), D. Donini (Italy), M. N. Ferencak, J. Grossman, N. Komanda, G. S. Lessells (Ireland), S. C. Locke, R. Martin, R. F. McCoart, S. Northshield, D. Raghavan, R. Stong, L. Zhou, GCHQ Problems Group (U. K.), NSA Problems Group, and the proposer.

Special Motzkin Paths

10816 [2000, 652]. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.* A *Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ with steps $(1, 1)$, $(1, 0)$, and $(1, -1)$ that never goes below the x-axis. For $n \ge 2$, show that the number of Motzkin paths of length n with no $(1, 0)$ steps on the x-axis is equal to the number of Motzkin paths of length $n-1$ with at least one $(1, 0)$ step on the x-axis.

Solution by Wen-jin Woan, Howard University, Washington, DC. We provide a bijection. Let P be a Motzkin path of length n with no $(1, 0)$ steps on the x-axis. Replace the first $(1, -1)$ step back to the x-axis with a $(1, 0)$ step and delete the first step of the path. We now have a Motzkin path Q of length $n-1$ with at least one $(1, 0)$ step on the x-axis. The new $(1, 0)$ step is the last $(1, 0)$ step on the x-axis of Q, and this identification provides the inverse mapping.

Editorial comment. Sen-Peng You noted the numbers counting these paths are studied under the name *Riordan numbers* in F. R. Bernhart, Catalan, Motzkin, and Riordan numbers, *Discrete Math.* 204 (1999) 73–112.

Solved also by M. H. Andreoli, M. Beck, D. Beckwith, M. A. P. Bernstein (France), J. C. Binz (Switzerland), M. Brozinsky, K. Calderhead, D. Callan, D. Cashing, N. Castaneda, R. J. Chapman (U. K.), P. P. Dalyay (Hungary), R. DiSario, D. Donini (Italy), T. Doslic (Croatia), J. Grossman, T. Kostic (Yugoslavia), K.-W. Lau (Hong Kong), G. S. Lessells (Ireland), K. Levasseur, O. P. Lossers (The Netherlands), K. N. Manoj (India), R. Martin,