SUMMER 2017 - COMBINATORICS 1 - TEST 1A - Solutions

A and B are finite sets with cardinalities a and b, respectively. $S(k,n)$ is a Stirling partition number. $p_n(k)$ is an integer partition number where n is the exact number of parts

1) $\sum_{n\geq 0} \alpha_n x^n$ is an ordinary generating function - *True, for the sequence* $\{a_n\}_{n\in\mathbb{N}}$

2) The generating function for the multichoose numbers is $(1+x^{-n})$ - *False,* $(1-x)^{-n}$

3) $[[(1+x)^n]]_{x^k} = \binom{k}{n}$ - *False, the coefficient of* x^k *in the binomial expansion is* $\binom{n}{k}$

4) $\left(\binom{a}{b}\right) = \binom{a-b+1}{b}$ - False, should be $\binom{a+b-1}{b}$

5) If the elements of A are indistinct and the elements of B are distinct, then the number of arbitrary functions from B to A is a Bell number - *True, Row 3, Col 1, TFW*

6) If the elements of A are distinct and the elements of B are indistinct, then the number of surjective functions from B to A is a Stirling number - *False, Row 2, Col 3, TFW*

7) The series represented by an exponential generating function must converge for all values of the variable - *False, formal power series need not converge*

8) If the elements of A are indistinct and the elements of B are distinct, then the number of injective functions from B to A is a Stirling number - *False, Row 3, Col 2, TFW*

9) If the elements of A are indistinct and the elements of B are distinct, then the number of arbitrary functions from A to B is a Bell number - *False, Row 2, Col 1, TFW*

10) $(-2)_3 = 0$ - *False,* $(-2)_3 = (-2)(-3)(-4) = -24$

11) If the elements of A are distinct and the elements of B are distinct, then the number of surjective functions from B to A is $a!S(b,a)$ - *True, Row 1, Col 3, TFW*

12) If the elements of A are indistinct and the elements of B are distinct, then the number of injective functions from B to A is zero if $b < a$ - *False, Row 3, Col 2, TFW*

13) $\binom{14}{2,3,4,4} = 120$ - *False*, $2+3+4+4 = 13 < 14$

14) The highest power of x that could appear in the rook polynomial for a 5x5 board with the main diagonal declared inaccessible is 4 - *False, filling in subdiagonal and top right square gives* 5

15) Given $n \in \mathbb{N}$, there is a bijection between partitions of n with odd parts and partitions with distinct parts. - *True, Euler's Theorem*

16) Combinatorics is fun

17) A combinatorial argument enumerates the same objects two different ways to establish equality of two counting formulas. - *True by definition*

18) $\binom{n}{1} + \binom{n}{2} \cdots \binom{n}{n-1} + \binom{n}{n} = 2^n$ - *False, need* $\binom{n}{0}$

19) $p_n(n) - p_{n-1}(n) = 1$ - *I misstated what I wanted here, so this is a free point for everyone*

20) If boards $B_1$and B_2 are disjoint, then $r(x,B_1 \cup B_2) = r(x,B_1)r(x,B_2)$ - *True, they are independent*

21) $N_=(\emptyset) = \sum_{J\subseteq P} (-1)^{|J|} N_{\geq}(J)$, where J is a subset of all the properties P that a

collection of objects may have, is a correct statement of the Inclusion/Exclusion Formula - *True,* $J : J \subseteq P$ *means the same*

22) Given OGF's $\sum_{j\geq 0} a_j x^j$ and $\sum_{j\geq 0} b_j x^j$, their convolution is defined to be $\sum_{k\geq 0}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$ - *True - Cauchy product*

23) The generating function $(1 + x + x^2 + \cdots + x^k + \cdots)^n$ yields the multichoose numbers for selecting elements from a set of size n with unlimited repetition - *True, equivalent to* $(1-x)^{-n}$

24) If the elements of A are distinct and the elements of B are distinct, then the number of arbitrary functions from A to B is b^a - *True, Row 1, Col 1, TFW*

25) If the elements of A are distinct and the elements of B are distinct, then the number of injective functions from A to B is $(b)_a$ - *True, Row 1 Col 2, TFW*

SUMMER 2017 - COMBINATORICS 1 - TEST 1B - Solutions

1) There are four conference rooms in a hotel: the Rose Room, Lilac Room, Orchid Room, and Sunflower Room. You have 200 identical chairs. How many ways can you provide seating in the conference rooms if you must have multiples of 20 chairs in each room, and no room is vacant?

Reduce problem to allocating units of 20 chairs. We have 10 indistinct units to be mapped surjectively to 4 distinct recipients.This is the same as mapping 6 units to 4 distinct recipients arbitrarily. The statistic that does this is the multichoose number $\left(\binom{4}{6}\right) = \binom{9}{6} = 84$

2) Find a formula for $\left(\binom{n}{0}\right) + \left(\binom{n}{1}\right) + \cdots + \left(\binom{n}{k}\right)$. Hint: $(1-x)^{-n}$ is the OGF for the multichoose numbers. Write $(1-x)^{-n} = (1-x)^{-n+1}(1-x)^{-1}$ and look at the coefficient of x^k on both sides. On the right hand side, you need to do a (trivial) convolution that produces the sum of multichoose numbers. Adjust summation indices as necessary to give the result.

This problem is worked out in detail on page 118 of the text. We also did the analogous problem with binomial coefficients in the test review....it resulted in Pascal's identity.