SPRING 2017 - CALCULUS 3 - TEST #1B - Solutions

Do **one** problem in class....no references, calculator only. Then do the remaining **four** at

home, all references permitted but no collaboration (God is watching). Due first class after break.

1) (Plane) A plane is determined by the three points $(1,2,3)$, $(-1,2,2)$, and $(3,0,1)$. What is the distance of this plane to the origin?

Strategy: Use the three points to get two vectors, then cross them to get a normal to their common plane. Use coordinates of one point to locate plane in space, then dot unitized normal with vector from origin to some point on plane. So: $P_1 = (1,2,3)$, $P_2 = (-1,2,2)$*, and* $P_3 = (3,0,1)$*. Then define* $A = \langle -1-1, 2-2, 2-3\rangle = \langle -2,0,-1\rangle$ *and* $B = \langle 3-1, 0-2, 1-2\rangle = \langle 2,-2,-2\rangle$*. These are in the plane. So the direction vector for the plane is* $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -1 \\ 2 & -2 & -2 \end{bmatrix} = -2i - 6j + 4k = 2\langle -1,-3,2\rangle$*. The unit direction vector is* $\frac{\langle -1,-3,2\rangle}{\sqrt{14}}$*. Now the vector* $\langle 3,0,1\rangle$ *runs from the origin to the point* $(3,0,1)$*, so dot this with the unit direction vector:* $\frac{\langle -1,-3,2\rangle}{\sqrt{14}} \bullet \langle 3,0,1\rangle = \frac{1}{3}(-3+2) = \frac{-1}{\sqrt{14}}$*. The absolute value is* $\frac{1}{\sqrt{14}}$ *. This is the projection of the vector from origin to plane in the direction of the normal to the plane...i.e. perpendicular distance.*

2) (Angle) Given the plane determined by the three points $(1,2,3)$, $(-1,2,2)$, and $(3,0,1)$ and another plane with equation $2x - 5y + 7z = 10$, determine if the planes intersect, and if so, at what (dihedral) angle.

We need a direction vector for the plane determined by the three points, so if we did (1) correctly, we already have it: $\langle -1,-3,2\rangle$*. The direction vector for the other plane is* $\langle 2,-5,7\rangle$*, and since these vectors are not scalar multiples of one another, they are not parallel (or antiparallel), hence the planes are not parallel either...i.e. they intersect. The usual formula finds the angle between direction vectors, which is the dihedral angle between planes.*

$\theta = \arccos\left(\frac{\langle -1,-3,2\rangle \bullet \langle 2,-5,7\rangle}{\|\langle -1,-3,2\rangle\| \|\langle 2,-5,7\rangle\|}\right) = \arccos\left(\frac{-2+15+14}{\sqrt{14}\sqrt{78}}\right) = 35.2$ *degrees*

3) (Plane) Determine c so that the three vectors $\langle c,4,-7\rangle$, $\langle 2,-1,4\rangle$, and $\langle 0,-9,18\rangle$ all lie in the same plane.

The solid figure determined by the three vectors would have zero volume if the three vectors

were coplanar. So we evaluate $\begin{bmatrix} c & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{bmatrix} = c(-18+36) - 4(36-0) - 7(-18-0) = 0$*.*
This simplifies to $18c - 144 + 126 = 0$*, so* $c = 1$*.*

4) (Arc length) Find the distance along the parabola $y = x^2$ from -1 to 1

*This is **not** the area under the curve...I assume everyone can do that integral easily. We represent this curve in* $\mathbb{R}^2$ *parametrically as* $\mathbf{r}(x) = \langle x, x^2 \rangle$*. We could have used* t*, but the parameter is just a dummy variable. Then* $\mathbf{r}'(x) = \langle 1, 2x \rangle$*. Recall the arclength formula:* $s = \int \|\mathbf{r}'(x)\| dx$*. By symmetry, we can find the arclength from* $x = 0$ *to* $x = 1$ *and double it. So* $s = 2\int_0^1 \sqrt{1+4x^2}\, dx$*. Using the trig substitution* $\tan\theta = 2x$*, we get* $\sec^2\theta d\theta = 2dx$ *and* $\sqrt{1+4x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$*. The integral simplifies to* $\int_0^{\arctan 2} \sec^3\theta d\theta \approx 2.958$*. By comparison, if we had just connected the starting and finishing points of the curve directly to the origin, the overall path length would have been* $2\sqrt{2} \approx 2.828$*, so we have confidence in the reasonableness of our answer.*

5) (Differential Geometry) A particle travels in a plane $(z = 0)$ elliptical path with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

This is a planar figure, so we know immediately that $\mathbf{B} = \mathbf{k}$*.The usual parametric*

representation for this ellipse is $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$*, therefore* $\mathbf{r}'(t) = \langle -2\sin t, 3\cos t \rangle$*.* $\mathbf{T}(t) = \frac{\langle -2\sin t, 3\cos t \rangle}{\|\langle -2\sin t, 3\cos t \rangle\|} = \frac{1}{\sqrt{4\sin^2 t + 9\cos^2 t}} \langle -2\sin t, 3\cos t \rangle$*. Since we are going to have to differentiate this again, we can make life a little simpler by writing* $\sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}$*. Now since* $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$*, we have*
$\mathbf{T}'(t) = \left[(4+5\cos^2 t)^{-1/2} \langle -2\sin t, 3\cos t \rangle \right]' =$
$(4+5\cos^2 t)^{-1/2} \langle -2\cos t, -3\sin t \rangle + \frac{1}{2}(4+5\cos^2 t)^{-3/2}(10\cos t\sin t)\langle -2\sin t, 3\cos t \rangle$*. Yeah...it's a mess by now, but...we can force a factor of* $(4+5\cos^2 t)^{-3/2}$ *out front by rewriting everything as* $(4+5\cos^2 t)^{-3/2}[(4+5\cos^2 t)\langle -2\cos t, -3\sin t \rangle + 5\cos t\sin t\langle -2\sin t, 3\cos t \rangle]$*, and combining the vectors we get* $(4+5\cos^2 t)^{-3/2}\langle A, B \rangle$*, where* $A = -8\cos t - 10\cos^3 t - 10\cos t\sin^2 t$ *and* $B = -12\sin t - 15\cos^2 t\sin t + 15\cos^2 t\sin t$*. I have to write them like this or the printer will crop the math. Now we see some simplification.* $A = -8\cos t - 10\cos t(\cos^2 t + \sin^2 t)$*, which reduces to* $-18\cos t$ *and* B *reduces immediately to* $-12\sin t$*. So*
$\mathbf{T}'(t) = (-6)(4+5\cos^2 t)^{-3/2}\langle 3\cos t, 2\sin t \rangle$*. Finally we have*

$\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \dfrac{\langle -3\cos t, -2\sin t\rangle}{\sqrt{4+5\cos^2 t}} = \mathbf{N}(t)$*. We have some confidence in our result since* $\mathbf{T}(t)\cdot\mathbf{N}(t) = \dfrac{1}{4+5\cos^2 t}[\langle -2\sin t, 3\cos t\rangle \cdot \langle -3\cos t, -2\sin t\rangle] = 0$*, so they are orthogonal.*

Note:
Here is a shortcut in $\mathbb{R}^2$. A unit vector in $\mathbb{R}^2$ has exactly two unit vectors perpendicular to it, unlike $\mathbb{R}^3$, where there are infinitely many. So if $\mathbf{T}(t) = \langle \alpha, \beta\rangle$, then $\mathbf{N}(t)$ would have to be either $\langle -\beta, \alpha\rangle$ or $\langle \beta, -\alpha\rangle$. Which one? The one that points to the center of curvature. For example, the unit circle is $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, so $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$. This also happens to be $\mathbf{T}(t)$, so possible $\mathbf{N}(t)$'s are $\langle -\cos t, -\sin t\rangle$ and $\langle \cos t, \sin t\rangle$. We know $\mathbf{r}''(t)$ points to the center of the circle, so $\mathbf{N}(t)$ would have to as well, and that means $\mathbf{N}(t) = \langle -\cos t, -\sin t\rangle$. Using that in our problem, $\mathbf{r}''(t) = \langle -2\cos t, -3\sin t\rangle$, which, for $t = 0$, for example, points directly at the origin (center of ellipse), so from $\mathbf{T}(t) = \dfrac{1}{\sqrt{4\sin^2 t + 9\cos^2 t}}\langle -2\sin t, 3\cos t\rangle$, we could immediately have said $\mathbf{N}(t) = \dfrac{1}{\sqrt{4\sin^2 t + 9\cos^2 t}}\langle -3\cos t, -2\sin t\rangle$